Public Provision of a Private Good as a Constrained Efficient Outcome

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October 4, 2005

Abstract

This paper shows that public provision of private goods may be justified on pure efficiency grounds in an environment where consumers consume both public and private goods. The idea is that public provision of a bundle that consists of a private and a public good good can make it easier to extract revenue from the customers for reasons familiar from the literature on multiproduct monopolies. We show that *partial* public provision of the private good improves economic efficiency under a condition that is always fulfilled under stochastic independence and satisfied for almost all distributions.

Keywords: Publicly Provided Private Goods; Constrained Efficiency.

JEL Classification Number: D61, D82, H42.

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1 Introduction

Public provision of private goods is quantitatively important in all developed countries. Economists usually think about either political economy explanations or redistributional concerns when trying to understand this phenomenon. A well-known example of a political economy rationale is Epple and Romano [7] who consider a model where the level of publicly provided "health services" is determined by the preferences of the median voter. If the median voter has an income that is below the mean income, there are positive transfers in equilibrium. The explanation is simply that the median voter obtains a net subsidy from richer citizens in this case. While political economy as in the Epple and Romano model are by no means implausible, one should notice that such models hinge crucially on the choice of policy instruments. That is, it is not explained why transfers must be in-kind, as opposed to pure cash transfers.[1] A related model is considered in Gouveia [9].

Another perspective is the one taken in Besley and Coate [1]. In their model, they note that some publicly provided goods are discrete in nature. In particular, if a household is dissatisfied with the quality of public schooling it is not always easy to "add-on" by buying a few extra high quality school hours from a private provider. Hence, the decision to attend public or private school is largely a binary decision. The point with their paper is that, if mainly rich households decide to opt out from public schooling, then the public school system becomes a transfer towards less wealthy individuals. Other arguments based on a desire to redistribute income can be found in Blomquist and Christensen [3], Cremer and Gahvarib [6].

The main purpose of this paper is to provide an alternative explanation of public provision of private goods based on ideas familiar from the literature on commodity bundling. That is, our starting point is that governments provide *both* purely private goods (such as public schooling and health care) and goods that are more or less non-rival (such as roads, parks, clean air protection, police and fire departments). Hence, it is natural to consider the question as to whether or not to socialize a private good in an economy where the government also needs to provide a public good.

In our model there are neither political economy or redistributional concerns. A benevolent planner seeks to maximize the social surplus (or, equivalently, the ex ante expected utility of a representative individual) in an economy with a binary and excludable public good, a binary private good, and a numeraire perfectly divisible private good. Agents are privately informed about their

[1] Coate [4] considers a model where in-kind redistribution may be better than cash transfers due to a time inconsistency problem that arises becasuse of altruistic preferences towards poor people. However, the median voter must expect to make ex post donations for this to explain a preference for in-kind transfers.

valuations of the binary private good and the public good.

We interpret outcomes where the price of the private good depends on whether or not the public good is consumed as *public provision* of the private good. The rationale for this interpretation is that in order to implement such a policy, it is necessary that the planner keeps track of which individuals purchase the private good and somehow tax them on the basis of whether the public good is consumed or not. One way of achieving this is to allow all consumers who opt in to the "government bundle" access to both the private good and the public good at no additional charge, much like paying property taxes allow citizens to enjoy the services of the local government in the US.

As a benchmark, we consider the case when, say, because of an inability to keep track individual purchases of the private good, the provision mechanism must be separable in the private and the public good. In this case the mechanism design problem simplifies to a problem where the planner needs to set an access fee for the public good, a provision probability of the public good, and a price for the private good. The solution to this problem has a flavor of Ramsey pricing. Not surprisingly, the price of the private good should always be set above the marginal cost of production. The idea is simply that the public good will be underprovided, so a small tax on the private good leads to a second order inefficiency in the market for the private good, and a first order efficiency gain in terms of public good provision. Maybe less expected and more interesting is that a similar logic implies that there should always be a strictly positive access fee, and that the public good should always be provided with *some probability*, whenever it is desirable from a first best efficiency point of view.

When allowing the mechanism to be nonseparable, it is no longer possible to get an analytic characterization of the optimal mechanism. Instead, we follow McAffee et al [13] and consider simple pricing policies, where the planner can select three prices-one for the private good, one for the public good, and one for the bundle consisting of both the goods. Our main result is that the solution to this problem is different from the best separable mechanism under a condition that is implied by stochastic independence between the valuation for the two goods, and is satisfied for virtually any joint distribution.[2] We thus conclude that, while we don't know the solution to the full mechanism design problem, we know that the solution must involve pricing of the private good in a way that is inconsistent with simply taxing purchases of the private good. Hence, we have a

[2] Truth in advertising: the claim that the condition is generic is at present a (well-founded) conjecture that we are currently trying to prove.

pure efficiency rationale for the public provision of private goods.[3]

We finally note that our paper is an example of an application where it is *necessary* to break the usual alignment of public economics into subfields that are considered separate problems. In most of the literature, public sector pricing, optimal taxation, and public goods provision are analyzed as if there is no connection between the problem. The public sector pricing problem formulated in the literature asks how to use indirect taxes and public sector prices to finance a given level of public expenditure, while the theory of public goods provision is concerned with how to elicit preferences in order to attain desirable outcomes. In contrast, this paper considers a problem with a non-trivial provision problem, as well as elements of public sector pricing. As a result, we obtain a novel explanation of why the public sector provides private goods.

2 The Environment

Consider an economy populated by a continuum of ex ante identical consumers. Consumers have preferences over a binary and excludable public good, a binary private good and a perfectly divisible private good that we will refer to as "money." The public good can be produced at a cost $K > 0$; and the binary private good which can be produced at unit cost $c > 0$.

A consumer is characterized by a valuation $\theta \in \Theta$ for the public good and a valuation $v \in V$ for the private good. A consumer's valuations (θ, v) is her private information. We denote $F : \Theta \times V \to [0,1]$ as the cumulative distribution over consumer types (θ, v) . We write $F_\theta : \Theta \to [0,1]$ and $F_v : V \to [0,1]$, respectively, as the marginal cumulative distribution over θ and v. Consumers are assumed to be risk neutral. The expected payoff of consumer of type (θ, v) who consumes the public good with probability ϕ and the private good with probability π and gives up (in expectation) m units of money is given by

$$u(\phi, \pi, m; \theta, v) = \phi\theta + \pi v - m. \tag{1}$$

The reader may observe that we have built in the assumption that wealth constraints are always non-binding in (1). If $\overline{\theta}$ and $\overline{v}$ are upper bounds for θ and v respectively, this assumption can be justified if every consumer has an endowment of money in excess of $\overline{\theta} + \overline{v}$.

[3]There are some efficiency based arguments in the literature, but they are either based on a limited set of policy instruments (Blackorby and Donaldson [2]), or time consistency considerations (Coate [4]).

2.1 Technical Remarks

We have chosen to model the consumers as a continuum in order to obtain a clean characterization of the optimal mechanism. The issue in a finite economy is that because first best efficiency is unattainable, the constrained efficient mechanism is a rather complicated object. In particular, the optimal provision rule depends in a non-trivial way on the realized types.[4] However, essentially due to a version of the paradox of voting, the effects of making the provision rule depend on announcements become negligible in a finite economy with sufficiently many consumers. As a result, Norman [16] shows that the optimal mechanism can be approximated by a mechanism where no consumer is pivotal to the decision.[5] Hence, our convenient continuum assumption can be viewed as looking at an approximation of a large finite economy.[6]

Indeed, the interpretation as an approximation is our prefered one. The reason is that the continuum specification has a few awkward technical aspects. In particular, we treat the distribution F as a cross section of consumers (we view the right hand side of the resource constraint (BB) below as the certain total revenue from the mechanism). Simultaneously, F is interpreted as the true probability distribution over types facing any individual consumer. As is well-known, this is inconsistent with stochastic independence across consumers in standard probability measures (see for example Judd [12]). But, dropping stochastic independence changes the nature of the problem (it opens up possibilities for first best efficiency), so taking he continuum as anything else but an approximation of a large finite economy is problematic..

3 Decentralized Provision of the Private Good

In this section we analyze the benchmark case where the mechanism designer cannot condition either the payment for or the provision probability of one good on her reported valuation for the other good. There are several ways to interpret this setup. The most obvious is simply that the

[4] See Cornelli [5], who points this out in the context of a profit maximizer with fixed cost of production. As shown in Norman [16], the surplus maximization problem is essentially a problem where a weighted average of profits and surplus is maximized, implying that the logic generalizes.

[5] See also Schmitz [20] who makes essentially the same point in the context of monopolistic provision of an excludable public good.

[6] Note however that to get a limit characterization corresponding to the one in this paper it is necessary to assume that per capita provision costs stay bounded away from zero as the number of participants goes out of bounds. If not, a pivot mechanism will work also with a large finite set of agents (see Hellwig [11]).

"markets" for the private and public goods are physically separated in space and that the designer lacks the technology to track behavior of individual consumers across markets. Formally:

Definition 1 *A separable mechanism is a quadruple* (ϕ, t, π, p), *where* $\phi : \Theta \rightarrow [0,1]$ *is the probability of consuming the public good and* $t : \Theta \rightarrow \mathrm{R}$ *is the fee for consuming the public good (both functions of the valuation for the public good only) and* $\pi : V \rightarrow [0,1]$ *is the probability of consuming the private good and* $p : V \rightarrow \mathrm{R}$ *is the fee for consuming the private good (both functions of the valuation of the private good only).*

Our notion of separability leaves room for cross-subsidization between the private and public goods. Such cross-subsidies are however the *only* link allowed between the two problems for separable mechanisms.

3.1 The Planning Problem

Given a separable mechanism, the expected utility for a consumer of type (θ, v) is given by $\phi(\theta)\theta - t(\theta) + \pi(v)v - p(v)$. We assume that the planner seeks to maximize the ex ante utility of the representative consumer, which may be written as,

$$\begin{aligned} & \int_\Theta \int_V [\phi(\theta)\theta - t(\theta) + \pi(v)v - p(v)]\, dF(\theta, v) \\ = & \int_\Theta [\phi(\theta)\theta - t(\theta)]\, dF_\theta(\theta) + \int_V [\pi(v)v - p(v)]\, dF_v(v). \end{aligned} \tag{2}$$

Since types are assumed to be private information consumers must be willing to disclose their preferences to the planner. That is, it must be *incentive compatible* to report truth-fully,

$$\phi(\theta)\theta - t(\theta) + \pi(v)v - p(v) \geq \phi(\widehat{\theta})\theta - t(\widehat{\theta}) + \pi(\widehat{v})v - p(\widehat{v}) \quad \forall (\theta, v), (\widehat{\theta}, \widehat{v}) \in \Theta \times V. \tag{IC}$$

We also assume that consumers must be willing to participate. Given the continuum-consumer formulation there is no distinction between interim and ex post participation constraints, and assuming that the non-participation utility is constant (and normalized to zero), we may write these constraints as

$$\phi(\theta)\theta - t(\theta) + \pi(v)v - p(v) \geq 0 \quad \forall (\theta, v) \in \Theta \times V. \tag{IR}$$

Finally, we assume that the planner must satisfy the natural social feasibility constraint, namely that the total costs for the production of the public and the private goods should not exceed the total revenue collected from the consumers. We may write this constraint as

$$K\left[\sup_{\theta\in\Theta}\phi(\theta)\right]+\int_V c\pi(v)\,dF_v(v)\leq\int_\Theta t(\theta)\,dF_\theta(\theta)+\int_V p(v)\,dF_v(v)\,. \tag{BB}$$

To understand (BB), note that, since the public good is non-rival, the cost is independent of the number of consumers actually consuming the good. That is, it costs K if the good is provided and 0 otherwise. Our formulation allows the mechanism designer to randomize between provision and non-provision. While (BB) says that resources balance in expectation, it is – once the constrained optimal mechanism is constructed – an easy matter to make sure that the budget is balanced with probability one even if it involves non-trivial randomizations.

3.2 Solving the Planning Problem

The first observation to make is that, which is the point with the separability assumptions, a mechanism is incentive compatible if and only if

$$\phi(\theta)\theta-t(\theta) \;\geq\; \phi(\widehat{\theta})\theta-t(\widehat{\theta}) \quad \forall\theta,\widehat{\theta}\in\Theta \tag{3}$$

$$\pi(v)v-p(v) \;\geq\; \pi(\widehat{v})v-p(\widehat{v}) \quad \forall v,\widehat{v}\in V. \tag{4}$$

That is, (3) can be viewed as the incentive compatibility constraint for the public good provision and (4) is the incentive compatibility constraint for the private good. Hence, the separability restrictions makes the characterization of incentive compatibility a single dimensional problem. As a result, the model is equivalent with a model with two sets of consumers, some who care only for the public good and others who only cares about the private good.

We now observe that if (ϕ^*,t^*,π^*,p^*) is an optimal separable mechanism, then

$$(\phi^*,t^*) \in \arg\max_{(\phi,t):\Theta\to[0,1]\times\mathrm{R}}\int_\Theta[\phi(\theta)\theta-t(\theta)]\,dF_\theta(v)+\int_V[\pi^*(v)v-p^*(v)]\,dF_v(v)\,. \tag{5}$$

$$\text{s.t. } 0 \leq \phi(\theta)\theta-t(\theta)-\phi(\widehat{\theta})\theta+t(\widehat{\theta}) \quad \forall\theta,\widehat{\theta}\in\Theta \tag{6}$$

$$0 \leq \phi(\theta)\theta-t(\theta)+\pi^*(v)v-p^*(v) \quad \forall(\theta,v)\in\Theta\times V \tag{7}$$

$$0 \leq \int_V t(\theta)\,dF_\theta(\theta)+\int_V p^*(v)\,dF_v(v)-K\left[\sup_{\theta\in\Theta}\phi(\theta)\right]-\int_V c\pi^*(v)\,dF_v(v) \tag{8}$$

and

$$
\begin{aligned}
(\pi^*, p^*) &\in \arg \max_{(\pi,p):V\to[0,1]\times\mathrm{R}} \int_\Theta \left[\phi^*(\theta)\,\theta - t^*(\theta)\right] dF_\theta(v) + \int_V \left[\pi(v)\,v - p(v)\right] dF_v(v)\,. && (9)\\
\text{s.t. } 0 &\le \pi(v)\,v - p(v) - \pi(\widehat{v})v + p(\widehat{v}) \quad \forall v, \widehat{v} \in \Theta && (10)\\
0 &\le \phi^*(\theta)\,\theta - t^*(\theta) + \pi(v)\,v - p(v) \quad \forall(\theta, v) \in \Theta \times V && (11)\\
0 &\le \int_V t^*(\theta)\,dF_\theta(\theta) + \int_V p(v)\,dF_v(v) - K\left[\sup_{\theta\in\Theta}\phi^*(\theta)\right] - \int_V c\pi(v)\,dF_v(v)\,. && (12)
\end{aligned}
$$

Both (5) and (9) are problems that can be solved using standard techniques going back to Myersons [14] analysis of optimal auction design. Define the "indirect utility functions"

$$
\begin{aligned}
U(\theta) &\equiv \theta\phi(\theta) - t(\theta) && (13)\\
W(v) &\equiv v\pi(v) - p(v)
\end{aligned}
$$

A routine argument can be used to establish the following lemma:

Lemma 1 *Suppose that $\Theta = \left[\underline{\theta}, \overline{\theta}\right]$ and that the marginal density $f_\theta(\theta)$ is strictly positive on its support. Then, (ϕ, t) satisfies the incentive compatibility constraints (6) if and only if ϕ is weakly increasing in θ and*

$$U(\theta) = U(\widehat{\theta}) + \int_{\widehat{\theta}}^{\theta} \phi(x)\,dx \quad \forall \theta, \widehat{\theta} \in \Theta.$$

Lemma 2 *Suppose that $V = [\underline{v}, \overline{v}]$ and that the marginal density $f_v(v)$ is strictly positive on its support. Then, (π, p) satisfies the incentive compatibility constraints (10) if and only if π is weakly increasing in v and*

$$W(v) = W(\widehat{v}) + \int_{\widehat{v}}^{v} \pi(x)\,dx \quad \forall v, \widehat{v} \in V.$$

Equally routine procedures, using Lemmas 1 and 2, show that the aggregate transfer revenues from the public goods fees and the private goods fees respectively can be determined purely in terms of the utility of the lowest type and the provision rules as

$$
\begin{aligned}
\int_\Theta t(\theta)\,dF_\theta(\theta) &= \int_\Theta \phi(\theta)\left(\theta - \frac{1 - F_\theta(\theta)}{f_\theta(\theta)}\right) dF_\theta(\theta) - U(\underline{\theta}) && (14)\\
\int_V p(v)\,dF_v(v) &= \int_V \pi(v)\left(v - \frac{1 - F_v(v)}{f_v(v)}\right) dF_v(v) - W(\underline{v}). && (15)
\end{aligned}
$$

We also observe that we without loss of generality may assume that the participation constraint of type $(\underline{\theta}, \underline{v})$ binds. That is,

Lemma 3 *Suppose that $(\phi^*, t^*, \pi^*, p^*)$ is an optimal separable mechanism. Then there exists $\left(\widetilde{t}, \widetilde{p}\right)$ such that $\left(\phi^*, \widetilde{t}, \pi^*, \widetilde{p}\right)$ is an optimal separable mechanism and*

$$\phi^*\left(\underline{\theta}\right)\underline{\theta} - \widetilde{t}\left(\underline{\theta}\right) + \pi^*\left(\underline{v}\right)\underline{v} - \widetilde{p}\left(\underline{v}\right) = 0$$

Since all higher types can mimic $\left(\underline{\theta}, \underline{v}\right)$, incentive compatibility automatically implies that the participation constraints hold for all higher types, provided that it is satisfied for type $\left(\underline{\theta}, \underline{v}\right)$. Using (14) and Lemma 3 we can therefore reformulate (5) as

$$\max_{\phi:\Theta\to[0,1]} \int_\Theta \left[\phi(\theta)\theta - K\phi\left(\overline{\theta}\right)\right] dF_\theta(v) \tag{16}$$

$$\text{s.t. } 0 \leq \phi(\theta)\left(\theta - \frac{1-F_\theta(\theta)}{f_\theta(\theta)} - K\phi\left(\overline{\theta}\right)\right) dF_\theta(\theta) + \int_V \left(p^*(v) - c\right) dF_v(v) \tag{17}$$

$$0 \leq \phi(\theta) \leq 1 \text{ for all } \theta \tag{18}$$

$$\phi \text{ is weakly increasing} \tag{19}$$

To understand the objective function, observe that the social feasibility constraint (12) must bind. The objective function (16) is thus simply obtained by substitution of (12) into the objective function of the problem, eliminating the constants, and noting, by the fact that ϕ is monotonic, that $\sup_{\theta\in\Theta}\phi(\theta) = \phi\left(\overline{\theta}\right)$. The integral constraint (17) together with the monotonicity requirement (19) combines all incentive and participation constraints, and the boundary constraints in (18) are just constraining the probabilities of provision to be probabilities.

In terms of further interpretations of the problem, it may be useful to observe that the problem for a profit maximizing monopolist would be to maximize

$$\int_\Theta \phi(\theta)\left(\theta - \frac{1-F_\theta(\theta)}{f_\theta(\theta)} - K\phi\left(\overline{\theta}\right)\right) dF_\theta(\theta) \tag{20}$$

subject only to the constraints (18) and (19). For this problem, the "no-haggling" logic of Myerson [14], Riley and Zeckhauser [17] and Stokey [21] immediately implies that the profit maximizing mechanism is, without loss of generality, one where the monopolist charges a single price.[7] However, without further constraints, this result does not extend to problems where profits appear as a constraint. In general, the solution to the problem (16) may be a randomized mechanism.[8] However,

[7] That is, if there is a profit maximizing random mechanism, then a single price mechanism that charges any price in the support of the randomized mechanism is also optimal.

[8] This is easy to realize by considering the case with two types, θ_L and θ_H. Assuming that charging a flat fee equal to θ_L would violate the budget constraint, whereas charging θ_H would give a strict surplus, it is obvious that the surplus can be made higher by letting the low type agents consume with some probability. The example can easily be extended to continuous densities.

randomizations can be ruled out by making some restrictions on the distribution of types. Define

$$x_\theta(\theta) \equiv \theta - \frac{1 - F_\theta(\theta)}{f_\theta(\theta)}, \tag{21}$$

which is often referred to as the "virtual surplus." We can then show the following result:

Proposition 1 *Suppose that $x_\theta(\theta)$ as defined in (21) is weakly increasing in θ and that ϕ^* is a solution to (16). Then there exists some t^* such that*

$$\phi^*(\theta) = \begin{cases} \phi\left(\overline{\theta}\right) & \text{for } \theta \geq \frac{t^*}{\phi(\overline{\theta})} \\ 0 & \text{for } \theta < \frac{t^*}{\phi(\overline{\theta})} \end{cases} \tag{22}$$

Hence, characterizing the solution to (16) is reduced to determining two variables: (1) the probability of provision $\phi\left(\overline{\theta}\right)$; and (2) a user fee (or equivalently, a threshold valuation for being allowed to consume the good when it is produced).

The continuum-consumer assumption in itself trivializes the provision decision in the sense that this can no longer be made contingent on the realized distribution of types. However, we still need to make a (standard) regularity assumption in order to obtain the fixed price characterization.[9]

In the same spirit, the private goods problem (9) may be reformulated as

$$\max_{\pi: V \to [0,1]} \int_V [\pi(v)v - p(v)]\, dF_v(v) \tag{23}$$

$$\text{s.t. } 0 \leq \int_V \pi(v)\left(v - \frac{1 - F_v(v)}{f_v(v)} - c\right) dF_v(\theta) + \int_\Theta t(\theta)\, dF_\theta(\theta) - K\phi\left(\overline{\theta}\right) \tag{24}$$

$$0 \leq \pi(v) \leq 1 \text{ for all } v \tag{25}$$

$$v \text{ is weakly increasing,} \tag{26}$$

and a similar argument allows us to conclude that all we need to do is to find a price to charge for the private good. Define

$$x_v(v) = v - \frac{1 - F_v(v)}{f_v(v)}. \tag{27}$$

Again, the fact that (23) is not a profit maximization problem makes it necessary to make regularity assumptions on the virtual surplus in order to rule out a randomized optimal mechanism. The result is;

[9] See Norman [16] for an assumption that justifies this particular continuum model as a limit of finite economies.

Proposition 2 *Suppose that $x_v(v)$ as defined in (27) is weakly increasing and that π^* is a solution to (23). Then, there exists some p^* such that*

$$\pi(v) = \begin{cases} 0 & \text{if } v < p^* \\ 1 & \text{if } v \geq p^* \end{cases} . \tag{28}$$

Propositions 1 and 2 show that, if the marginal distributions are such that the virtual surplus for each good is monotonic in type, the maximization of (2) subject to (IC),(IR) and (BB) reduces to a simple optimization problem in three variables: (1) a flat fee user t for the public good; (2) a probability that the public good is available ϕ (which may also be reinterpreted as a quantity); and (3) a fixed price p for the private good.

Hence, using Propositions 1 and 2 we obtain the following simplified planning problem:

$$\max_{\{t,\phi,p\}} \phi \int_{\frac{t}{\phi}}^{\overline{\theta}} \theta dF_\theta(\theta) - K\phi + \int_p^{\overline{v}} (v-c)\, dF_v(v) \tag{29}$$

$$\text{s.t } 0 \leq t\left(1 - F_\theta\left(\frac{t}{\phi}\right)\right) + p\left(1 - F_v(p)\right) - K\phi - c\left(1 - F_v(p)\right). \tag{30}$$

$$0 \leq \phi \leq 1 \tag{31}$$

Proposition 3 *Suppose that $\mathrm{E}\theta > K$ and that $\underline{\theta} < K$. Then, in any optimal solution (t^*, ϕ^*, p^*) to (29) the following is true;*

1. $p^* > c$;
2. $\phi^* > 0$;
3. $t^* > 0$.

Proof. The Kuhn-Tucker necessary conditions for an optimum are,

$$0 = \int_{\frac{t}{\phi}}^{\overline{\theta}} \theta dF_\theta(\theta) + f_\theta\left(\frac{t}{\phi}\right)\left(\frac{t}{\phi}\right)^2 - K + \lambda f_\theta\left(\frac{t}{\phi}\right)\left(\frac{t}{\phi}\right)^2 - \lambda K + \mu - \gamma \tag{32}$$

$$\mu\phi = 0, \gamma(1-\phi) = 0, \mu \geq 0, \gamma \geq 0 \tag{33}$$

$$0 = -\frac{t}{\phi} f_\theta\left(\frac{t}{\phi}\right) + \lambda\left[1 - F_\theta\left(\frac{t}{\phi}\right) - f_\theta\left(\frac{t}{\phi}\right)\frac{t}{\phi}\right] \tag{34}$$

$$0 = -(p-c)\, f_v(p) + \lambda\left[(1 - F_v(p)) - (p-c)\, f_v(p)\right] \tag{35}$$

$$0 = \lambda\left[t\left(1 - F_\theta\left(\frac{t}{\phi}\right)\right) + p\left(1 - F_v(p)\right) - K\phi - c\left(1 - F_v(p)\right)\right], \lambda \geq 0 \tag{36}$$

Part 1: If $p^* < c$ then the first term on the right hand side in (35) is strictly positive and the second is weakly positive, implying that the condition cannot hold. Suppose that $p^* = c$. Then, from (35) either $\lambda = 0$ or $(1 - F_v(c)) = 0$. Since the second condition is ruled out by the assumption that $c < \overline{v}$, the only possibility that remains is that $\lambda = 0$. But if $\lambda = 0$ at the optimal solution, then t^*, ϕ^* must solve

$$\max_{t,\phi} \phi \int_{\frac{t}{v}}^{\overline{\theta}} \theta dF_\theta(\theta) - K\phi$$

We note that if $\phi^* > 0$ in the solution, then the objective is monotonically decreasing in t over $\left[\underline{\theta}, \overline{\theta}\right]$, so either $\phi^* = 0$ or $\phi^* > 0$ and $t^* = \phi^* \underline{\theta}$. But, then ϕ^* must maximize

$$\phi \left[\int_{\underline{\theta}}^{\overline{\theta}} \theta dF_\theta(\theta) - K \right] = \phi \left[\mathrm{E}\theta - K \right].$$

By assumption the bracketed expression is strictly positive, so the solution must be $\phi^* = 1$. Since the associated surplus is strictly larger than zero (which is the surplus when $\phi = 0$), we conclude that this must indeed be the solution if $\lambda = 0$. But, substituting $p^* = c, t^* = \underline{\theta}$ and $\phi^* = 1$ into the constraint we see that

$$t^* \left(1 - F_\theta \left(\frac{t^*}{\phi^*} \right) \right) + p^* (1 - F_v(p^*)) - K\phi^* - c(1 - F_v(p^*)) = \underline{\theta} - K < 0.$$

Hence, the resource constraint is violated. It follows that $p^* > c$ in any solution to (29).

Part 2: Follows immediately from Part 1 since if $p^* > c$ and $\phi^* = 0$ there is a strict budget surplus. Fix $p^*, t^* = 0$ and let ϕ' be given by

$$\phi' = \frac{(p^* - c)(1 - F_v(p^*))}{K} > 0$$

By construction, the constraint is satisfied and the surplus under the alternative $\left(t^* = 0, \phi', p^*\right)$ is

$$\phi' \mathrm{E}\theta + \int_{p^*}^{\overline{v}} (v - c) dF_v(v) > \int_{p^*}^{\overline{v}} (v - c) dF_v(v).$$

Hence, $\left(t^* = 0, \phi', p^*\right)$ results in a strict increase in surplus relative $(t^* = 0, \phi^* = 0, p^*)$.

Part 3: This is obvious if $\underline{\theta} > 0$, since $t^* = \phi^* \underline{\theta}$ would be non-distortionary. Suppose $\underline{\theta} \leq 0$ and observe that the necessary condition (34) must be satisfied, that is

$$0 = \lambda \left[1 - F_\theta(0) \right].$$

This condition can only hold if $\lambda = 0$, but $p^* = c$, which contradicts our conclusion in Part 1. ■

3.3 Summary

In this section we postulated that in the provision of either the private or the public good, neither the probability that a consumer gets access to the good or the fee depends on the valuation for the other good. No other restrictions were made on the set of admissible mechanisms, but since the design problem under our separability assumptions can be solved using auxiliary single dimensional problems, the structure of any optimal mechanism is very simple. Under some standard regularity assumptions we only need to determine two prices, one for the public and one for the private good, and a probability to provide the public good. This simple structure justifies our labeling of this regime as "decentralized provision of the private good", since a per unit tax on the private good is the only intervention needed in the "private sector".

There is always a cross subsidy from the private good to the production of the public good at the optimum. This is hardly surprising. The logic is simply that the welfare cost of a small tax is second order, since the consumers who stop consuming the private good have valuations just barely above the cost of production. What we find more interesting is that a public project with positive expected value in case of no exclusions *should always be undertaken with some probability.* That is, there is a role for randomizations in the model. Moreover, we also find it interesting to observe that *there should always be a strictly positive user fee for the public good.* The logic is similar to the argument for why there should be a positive tax on the private sector, but runs counter the idea that excluding consumers when the marginal cost is zero is always bad.

4 Public Provision of Both Goods

The optimal mechanism in Section 3 is consistent with an economy where the private good is traded on a competitive market (subject to a tax), and where the public good is provided by a government entity with resources coming from user fees and tax revenue. We shall now consider a setup where the government is able to condition the provision probability and price for each of the two goods on the valuation for *both goods*. We interpret this as public provision since this is inconsistent with a world where the trading of the private good is done anonymously.

In general, a direct revelation mechanism can be represented as a quadruple $\left(\widetilde{\phi}, \widetilde{f}, \widetilde{\pi}, \widetilde{p}\right)$, where the difference with Section 3 is that all these functions are over the domain $\Theta \times V$, whereas the corresponding objects in the separable case are functions of either Θ or V. This leads to a multidimensional mechanism design problem, and there is no known methodology for how to characterize incentive compatibility in an analytically tractable way.

To get a tractable problem we will proceed along the lines of McAffe et al [13] and consider a simple class of mechanisms. Specifically, we will add a single instrument to the separable case, so that instead of considering mechanisms on the form (t, ϕ, p), we will consider mechanisms on the form (t, ϕ, p, τ), where (t, ϕ, p) have the same interpretations as before as user fee and provision probability for the public good and the price of the private but, and τ now is the fee charged for a consumer who consumes both the public and the private good. If $\tau \neq t + p$ this requires that the government is actively involved in provision of the private good.

While it is obviously a limitation that we are not able to characterize the constrained efficient mechanism for the full-blown mechanism design problem, the reader may note that, if we find that $\tau \neq t + p$ in the solution to our simplified problem, then it must be that the constrained efficient mechanism is also one in which the government takes an active part in the provision of the private good. This will therefore answer the qualitative question we are interested in; whether public provision of a private good can be efficiency enhancing or not.

4.1 Some Preliminaries

Consider a (simple pricing) mechanism on the form (t, ϕ, p, τ). A consumer will demand:

Only the Public Good if

$$\begin{aligned} \phi\theta - t &\geq 0 \\ \phi\theta - t &\geq v - p \\ \phi\theta - t &\geq \phi\theta + v - \tau \end{aligned} \tag{37}$$

Only the Private Good if

$$\begin{aligned} v - p &\geq 0 \\ v - p &\geq \phi\theta - t \\ v - p &\geq \phi\theta + v - \tau \end{aligned} \tag{38}$$

The Bundle if

$$\begin{aligned} \phi\theta + v - \tau &\geq 0 \\ \phi\theta + v - \tau &\geq \phi\theta - t \\ \phi\theta + v - \tau &\geq v - p \end{aligned} \tag{39}$$

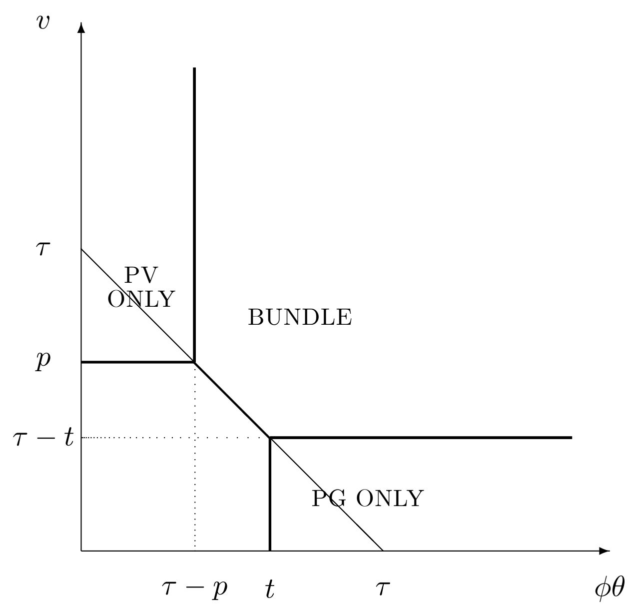

Figure 1:

Which of the inequalities in (37), (38) and (39) are relevant depends on whether the bundle is cheaper or more expensive than the components.

Claim 1 *If $\tau \leq t+p$, the second inequality in (37) and (38) is implied by the other two inequalities and the proportion of consumers purchasing;*

1. *the public good only is*
$$\int_{\frac{t}{\phi}}^{\overline{\theta}} \int_{\underline{v}}^{\tau-t} f(\theta, v)\, dv d\theta$$
2. *the private good only is*
$$\int_{p}^{\overline{v}} \int_{\underline{\theta}}^{\frac{\tau-p}{\phi}} f(\theta, v)\, d\theta dv$$
3. *the bundle is*
$$\int_{\frac{\tau-p}{\phi}}^{\frac{t}{\phi}} \int_{\tau-\phi\theta}^{\overline{v}} f(\theta, v)\, dv d\theta + \int_{\frac{t}{\phi}}^{\overline{\theta}} \int_{\tau-t}^{\overline{v}} f(\theta, v)\, dv d\theta$$

Claim 2 *If $\tau \geq t+p$, the first inequality in (39) is implied by the other two inequalities and and the proportion of consumers purchasing;*

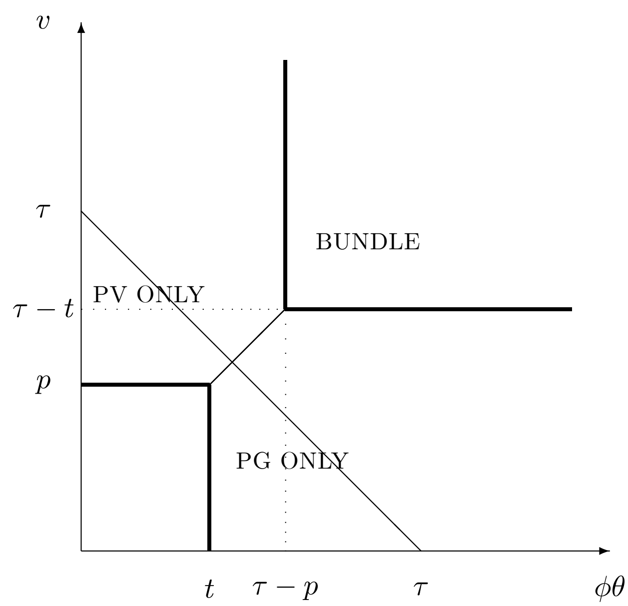

Figure 2:

1. *the public good only is*

$$\int_{\frac{t}{\phi}}^{\frac{\tau-p}{\phi}}\int_{\underline{v}}^{\phi\theta+p-t} f(\theta,v)\,dvd\theta+\int_{\frac{\tau-p}{\phi}}^{\overline{\theta}}\int_{\underline{v}}^{\tau-t} f(\theta,v)\,dvd\theta$$

2. *the private good only is*

$$\int_{p}^{\tau-t}\int_{\underline{\theta}}^{\frac{v+t-p}{\phi}} f(\theta,v)\,d\theta dv+\int_{\tau-t}^{\overline{v}}\int_{\underline{\theta}}^{\frac{\tau-p}{\phi}} f(\theta,v)\,d\theta dv$$

3. *the bundle is*

$$\int_{\frac{\tau-p}{\phi}}^{\overline{\theta}}\int_{\tau-t}^{\overline{v}} f(\theta,v)\,dvd\theta$$

We thus need to separate between two cases, one with $\tau \le t+p$ and one with $\tau \ge t+p$. We define by $G_1(t,p,\tau;\phi)$ the budget surplus (when positive) given a mechanism (t,ϕ,p,τ) where $\tau \le t+p$. That is

$$\begin{aligned} G_1(t,p,\tau;\phi) &= t\left[\int_{\frac{t}{\phi}}^{\overline{\theta}}\int_{\underline{v}}^{\tau-t} f(\theta,v)\,dvd\theta\right]+(p-c)\left[\int_{p}^{\overline{v}}\int_{\underline{\theta}}^{\frac{\tau-p}{\phi}} f(\theta,v)\,d\theta dv\right] \\ &\quad+(\tau-c)\left[\int_{\frac{\tau-p}{\phi}}^{\frac{t}{\phi}}\int_{\tau-\phi\theta}^{\overline{v}} f(\theta,v)\,dvd\theta+\int_{\frac{t}{\phi}}^{\overline{\theta}}\int_{\tau-t}^{\overline{v}} f(\theta,v)\,dvd\theta\right]-K\phi, \end{aligned} \tag{40}$$

Symmetrically, we let $G_2(t,p,\tau;\phi)$ denote the budget surplus given a mechanism (t,ϕ,p,τ) where $\tau \geq t+p$,

$$\begin{aligned} G_2(t,p,\tau;\phi) &= t\left[\int_{\frac{t}{\phi}}^{\frac{\tau-p}{\phi}}\int_{\underline{v}}^{\phi\theta+p-t} f(\theta,v)\,dvd\theta + \int_{\frac{\tau-p}{\phi}}^{\overline{\theta}}\int_{\underline{v}}^{p} f(\theta,v)\,dvd\theta\right] \\ &+(p-c)\left[\int_{p}^{\tau-t}\int_{\underline{\theta}}^{\frac{v+t-p}{\phi}} f(\theta,v)\,d\theta dv + \int_{\tau-t}^{\overline{v}}\int_{\underline{\theta}}^{\frac{\tau-p}{\phi}} f(\theta,v)\,d\theta dv\right] \\ &+(\tau-c)\left[\int_{\frac{\tau-p}{\phi}}^{\overline{\theta}}\int_{\tau-t}^{\overline{v}} f(\theta,v)\,dvd\theta\right] - K\phi. \end{aligned} \tag{41}$$

Next, let $S_1(t,p,\tau;\phi)$ denote the social surplus associated with (t,ϕ,p,τ) in the case when $\tau \leq t+p$,

$$\begin{aligned} S_1(t,p,\tau;\phi) &= \int_{\frac{t}{\phi}}^{\overline{\theta}}\int_{\underline{v}}^{\tau-t} \phi\theta f(\theta,v)\,dvd\theta + \int_{p}^{\overline{v}}\int_{\underline{\theta}}^{\frac{\tau-p}{\phi}} (v-c) f(\theta,v)\,d\theta dv \\ &+\int_{\frac{\tau-p}{\phi}}^{\frac{t}{\phi}}\int_{\tau-\phi\theta}^{\overline{v}} (\phi\theta+v-c) f(\theta,v)\,dvd\theta \\ &+\int_{\frac{t}{\phi}}^{\overline{\theta}}\int_{\tau-t}^{\overline{v}} (\phi\theta+v-c) f(\theta,v)\,dvd\theta - K\phi, \end{aligned} \tag{42}$$

and let $S_2(t,p,\tau;\phi)$ be the social surplus associated with (t,ϕ,p,τ) in the case when $\tau \geq t+p$,

$$\begin{aligned} S_2(t,p,\tau;\phi) &= \int_{\frac{t}{\phi}}^{\frac{\tau-p}{\phi}}\int_{\underline{v}}^{\phi\theta+p-t} \phi\theta f(\theta,v)\,dvd\theta + \int_{\frac{\tau-p}{\phi}}^{\overline{\theta}}\int_{\underline{v}}^{\tau-t} \phi\theta f(\theta,v)\,dvd\theta \\ &+\int_{p}^{\tau-t}\int_{\underline{\theta}}^{\frac{v+t-p}{\phi}} (v-c) f(\theta,v)\,d\theta dv + \int_{\tau-t}^{\overline{v}}\int_{\underline{\theta}}^{\frac{\tau-p}{\phi}} (v-c) f(\theta,v)\,d\theta dv \\ &+\int_{\frac{\tau-p}{\phi}}^{\overline{\theta}}\int_{\tau-t}^{\overline{v}} (\phi\theta+v-c) f(\theta,v)\,dvd\theta - K\phi. \end{aligned} \tag{43}$$

4.2 The Main Result

For notational brevity we will let $z=(t,p,\tau)$ and $z^*=(t^*,p^*,t^*+p^*)$, where (t^*,ϕ^*,p^*) is an optimal solution to the problem (29). Differentiating (40) and evaluating at $z=z^*$ we find that:

$$\begin{aligned} \frac{\partial G_1(z^*;\phi^*)}{\partial t} &= \int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} \{F_v(p^*|\theta)+(p^*-c)f_v(p^*|\theta)\} f_\theta(\theta)\,d\theta - \frac{t^*}{\phi^*}F_v\left(p^*|\frac{t^*}{\phi^*}\right) f_\theta\left(\frac{t^*}{\phi^*}\right) \\ \frac{\partial G_1(z^*;\phi^*)}{\partial p} &= \int_{\underline{\theta}}^{\frac{t^*}{\phi^*}} [\{(1-F_v(p^*|\theta))-(p^*-c)f_v(p^*|\theta)\}] f_\theta(\theta)\,d\theta + \frac{t^*}{\phi^*}\left[1-F_v\left(p^*|\frac{t^*}{\phi^*}\right)\right] f_\theta\left(\frac{t^*}{\phi^*}\right) \\ \frac{\partial G_1(z^*;\phi^*)}{\partial \tau} &= \int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} \{(1-F_v(p^*|\theta))-(p^*-c)f_v(p^*|\theta)\} f_\theta(\theta)\,d\theta - \frac{t^*}{\phi^*}\left[1-F_v\left(p^*|\frac{t^*}{\phi^*}\right)\right] f_\theta\left(\frac{t^*}{\phi^*}\right) \end{aligned} \tag{44}$$

The reader may notice that the second expression in (44) corresponds directly to the expression in Proposition 1 in McAffee et al [13]. The first equation can also be written in that form by reversing the roles of θ and v, but we have chosen to be consistent and always write the first expression as an integral with respect to θ. This close correspondence with McAffee et al is no coincidence. The derivatives reported above can be thought of as the effect on profits given a marginal increase in t, p and τ respectively, which is exactly what McAffee et al [13] is analyzing. In their case, going from (44) to their main result is relatively straightforward since they ask for a direction where the mixed bundling mechanism increases *profits* relative to separate provision. In particular, since (t^*, p^*) in their case would be chosen to solve a profit maximization problem, the term

$$\begin{aligned} & \int_{\underline{\theta}}^{\frac{t^*}{\phi^*}} \left[\left\{(1 - F_v\left(p^*|\theta\right)) - \left[p^* - c\right] f_v\left(p^*|\theta\right)\right\}\right] f_\theta\left(\theta\right) d\theta \\ = & \int_{\underline{\theta}}^{\frac{t^*}{\phi^*}} \left[\left\{(1 - F_v\left(p^*\right)) - \left[p^* - c\right] f_v\left(p^*\right)\right\}\right] f_\theta\left(\theta\right) = 0 \end{aligned} \tag{45}$$

if θ and v are stochastically independent, so it follows immediately from (44) that a small increase in the price of the private good or a small decrease in the price from the bundle would increase the profits in the case of stochastic independence. It is not as obvious from (44), but by rewriting $\partial G_1\left(z^*; \phi^*\right)/\partial t$ one can also check that a small increase in the price of the public good also increases profits if θ and v are stochastically independent.[10]

Our problem is different in two respects. First of all, we like to demonstrate that bundling can increase *economic efficiency* rather than profits. Secondly, because (t^*, p^*) are *not profit maximizing*, we cannot use the first order conditions from the best separable mechanism in the same way as McAffee et al [13].

Differentiating (42) and evaluating at $z = z^*$ we find that

$$\begin{aligned} \frac{\partial S_1\left(z^*; \phi^*\right)}{\partial t} &= \int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} \left(p^* - c\right) f_v\left(p^*|\theta\right) f_\theta\left(\theta\right) d\theta - \frac{t^*}{\phi^*} F_v\left(p^*|\frac{t^*}{\phi^*}\right) f_\theta\left(\frac{t^*}{\phi^*}\right) \\ \frac{\partial S_1\left(z^*; \phi^*\right)}{\partial p} &= -\int_{\underline{\theta}}^{\frac{t^*}{\phi^*}} \left(p^* - c\right) f_v\left(p^*|\theta\right) f_\theta\left(\theta\right) d\theta + \frac{t^*}{\phi^*}\left[1 - F_v\left(p^*|\frac{t^*}{\phi^*}\right)\right] f_\theta\left(\frac{t^*}{\phi^*}\right) \\ \frac{\partial S_1\left(z^*; \phi^*\right)}{\partial \tau} &= -\int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} \left(p^* - c\right) f_v\left(p^*|\theta\right) f_\theta\left(\theta\right) d\theta - \frac{t^*}{\phi^*}\left[1 - F_v\left(p^*|\frac{t^*}{\phi^*}\right)\right] f_\theta\left(\frac{t^*}{\phi^*}\right) \end{aligned} \tag{46}$$

Our first preliminary result is that, evaluated at an optimal solution to the problem (29), the partial derivatives of G_1 and G_2 are the same, and the partial derivatives of S_1 and S_2 also coincide.

[10] Verify...

Letting $DG_i(z;;\phi)$ and $DS_i(z;\phi)$ denote the gradient vectors for $i = 1,2$ we thus have that;

Lemma 4 *$DG_1(z^*;\phi^*) = DG_2(z^*;\phi^*)$ and $DS_1(z^*;\phi^*) = DS_2(z^*;\phi^*)$*

Now consider the following problem.

$$\begin{aligned} & \max_{(t,p,\tau)} S_1(t,p,\tau;\phi^*) \\ \text{s.t.} \quad & G_1(t,p,\tau;\phi^*) \geq 0 \\ & t + p - \tau \geq 0. \end{aligned} \tag{47}$$

The problem (47) gives the best simple pricing policy where the bundle is cheaper than its components when sold separately. Obviously, if the best simple pricing policy has this property, then (47) gives the optimal solution to the full problem. Symmetrically, the problem

$$\begin{aligned} & \max_{(t,p,\tau)} S_2(t,p,\tau;\phi^*) \\ \text{s.t.} \quad & G_2(t,p,\tau;\phi^*) \geq 0 \\ & \tau - t - p \geq 0, \end{aligned} \tag{48}$$

gives the best simple pricing policy where the bundle is more expensive than the components. Notice that $z^* = (t^*, p^*, t^* + p^*)$, the solution to the optimization problem where the markets had to be treated separately, problem (29), is in the constraint set of both (47) and (48).

We now observe the following;

Lemma 5 *Let λ^* be the multiplier on constraint (30) corresponding to the solution (ϕ^*, t^*, p^*) of problem (29). Also, let λ_i be the multiplier on the resource constraint $G_i(t,p,\tau;\phi^*)$ for $i = 1,2$ in problem (47) and (48). Then;*

1. *$\lambda_1 = \lambda^*$ if z^* solves problem (47);*
2. *$\lambda_2 = \lambda^*$ if z^* solves problem (48).*

Proof. First consider (47). If z^* solves the problem, the Kuhn-Tucker necessary conditions for a solution must be fulfilled at z^*. Hence, there must exist $\lambda_1 > 0$ and $\mu_1 \geq 0$ such that

$$\begin{aligned} & \frac{\partial S_1(z^*;\phi^*)}{\partial t} + \lambda_1 \frac{\partial G_1(z^*;\phi^*)}{\partial t} + \mu_1 = 0 \\ & \frac{\partial S_1(z^*;\phi^*)}{\partial p} + \lambda_1 \frac{\partial G_1(z^*;\phi^*)}{\partial p} + \mu_1 = 0 \\ & \frac{\partial S_1(z^*;\phi^*)}{\partial \tau} + \lambda_1 \frac{\partial G_1(z^*;\phi^*)}{\partial \tau} - \mu_1 = 0 \\ & \mu_1 (t + p - \tau) = 0, \quad \mu_1 \geq 0 \end{aligned} \tag{49}$$

Using the expressions for the partial derivatives in (44) and (46) it is easy to check that;

$$\begin{aligned} \frac{\partial S_1(z^*;\phi^*)}{\partial t}+\frac{\partial S_1(z^*;\phi^*)}{\partial \tau} &= -\frac{t^*}{\phi^*}f_\theta\left(\frac{t^*}{\phi^*}\right) \\ \frac{\partial G_1(z^*;\phi^*)}{\partial t}+\frac{\partial G_1(z^*;\phi^*)}{\partial \tau} &= \int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} f_\theta(\theta)\,d\theta-\frac{t^*}{\phi^*}f_\theta\left(\frac{t^*}{\phi^*}\right) \\ &= \left[1-F_\theta\left(\frac{t^*}{\phi^*}\right)\right]-\frac{t^*}{\phi^*}f_\theta\left(\frac{t^*}{\phi^*}\right). \end{aligned} \tag{50}$$

Combining the first and third condition in (49) and using (50) we have that

$$-\frac{t^*}{\phi^*}f_\theta\left(\frac{t^*}{\phi^*}\right)+\lambda_1\left\{\left[1-F_\theta\left(\frac{t^*}{\phi^*}\right)\right]-\frac{t^*}{\phi^*}f_\theta\left(\frac{t^*}{\phi^*}\right)\right\}=0. \tag{51}$$

This condition is the same as (34), the first order condition to the problem when the goods are sold separately. It follows that $\lambda_1=\lambda^*$, since otherwise (51) will be violated. This proves the first part.

For the second part, we note that the Kuhn-Tucker conditions are

$$\begin{aligned} \frac{\partial S_2(z^*;\phi^*)}{\partial t}+\lambda_2\frac{\partial G_2(z^*;\phi^*)}{\partial t}-\mu_2 &= \frac{\partial S_1(z^*;\phi^*)}{\partial t}+\lambda_2\frac{\partial G_1(z^*;\phi^*)}{\partial t}-\mu_2=0 \\ \frac{\partial S_2(z^*;\phi^*)}{\partial p}+\lambda_2\frac{\partial G_2(z^*;\phi^*)}{\partial p}-\mu_2 &= \frac{\partial S_1(z^*;\phi^*)}{\partial p}+\lambda_2\frac{\partial G_1(z^*;\phi^*)}{\partial p}-\mu_2=0 \\ \frac{\partial S_2(z^*;\phi^*)}{\partial \tau}+\lambda_2\frac{\partial G_2(z^*;\phi^*)}{\partial \tau}+\mu_2 &= \frac{\partial S_1(z^*;\phi^*)}{\partial \tau}+\lambda_2\frac{\partial G_1(z^*;\phi^*)}{\partial \tau}+\mu_2=0 \\ \mu_2(t+p-\tau) &= 0,\mu_2\geq 0 \end{aligned} \tag{52}$$

The same argument applies. ■

Together, Lemmas 4 and 5 makes the Kuhn-Tucker conditions for problem (47) comparable with those of problem (48). It now follows more or less directly;

Proposition 4 *Let λ^* be the multiplier on constraint (30) corresponding to the solution (ϕ^*,t^*,p^*) of problem (29). Then, there exists a simple pricing policy (t,p,τ) (involving provision of the private good bundled together with provision of the public good) that is feasible and generates a higher social surplus whenever*

$$DS_1(z^*;\phi^*)+\lambda^*DG_1(z^*;\phi^*)\neq 0.$$

Proof. From Lemma 5 we know that if z^* solves both problems (47) and (48), the multiplier in each problem must be given by λ^*. Thus if z^* is the best simple pricing policy for problem (47), then

$$\begin{aligned} \frac{\partial S_1(z^*;\phi^*)}{\partial t} + \lambda^* \frac{\partial G_1(z^*;\phi^*)}{\partial t} + \mu_1 &= 0 \\ \frac{\partial S_1(z^*;\phi^*)}{\partial p} + \lambda^* \frac{\partial G_1(z^*;\phi^*)}{\partial p} + \mu_1 &= 0 \\ \frac{\partial S_1(z^*;\phi^*)}{\partial \tau} + \lambda^* \frac{\partial G_1(z^*;\phi^*)}{\partial \tau} - \mu_1 &= 0 \\ \mu_1 (t + p - \tau) = 0, \qquad & \mu_1 \geq 0. \end{aligned} \tag{53}$$

Similarly if z^* is the best simple pricing policy for problem (48), then by using Lemma 4, we have

$$\begin{aligned} \frac{\partial S_1(z^*;\phi^*)}{\partial t} + \lambda^* \frac{\partial G_1(z^*;\phi^*)}{\partial t} - \mu_2 &= 0 \\ \frac{\partial S_1(z^*;\phi^*)}{\partial p} + \lambda^* \frac{\partial G_1(z^*;\phi^*)}{\partial p} - \mu_2 &= 0 \\ \frac{\partial S_1(z^*;\phi^*)}{\partial \tau} + \lambda^* \frac{\partial G_1(z^*;\phi^*)}{\partial \tau} + \mu_2 &= 0 \\ \mu_2 (t + p - \tau) = 0, \qquad & \mu_2 \geq 0 \end{aligned} \tag{54}$$

Assume that $\mu_1 > 0$. Then, (53) implies that $\frac{\partial S_1(z^*;\phi^*)}{\partial t} + \lambda^* \frac{\partial G_1(z^*;\phi^*)}{\partial t} < 0$, which makes it impossible to find $\mu_2 \geq 0$ such that (54) holds. Symmetrically, if $\mu_2 > 0$, then $\frac{\partial S_1(z^*;\phi^*)}{\partial t} + \lambda^* \frac{\partial G_1(z^*;\phi^*)}{\partial t} > 0$, which makes it impossible to find $\mu_1 \geq 0$ such that (53) holds. Since z^* must solve both (47) and (48) for there to be no improvement we conclude that $\mu_1 = \mu_2 = 0$, or else there is some z better than z^*. The claim follows. ∎

4.3 Stochastic Independence

The case with Stochastic independence is an important benchmark that deserves some special attention. In this case we have that there is indeed always an improvement over the best separate provision policy;

Proposition 5 *Suppose that* θ *and* v *are stochastically independent. Then* $\frac{\partial S_1(z^*;\phi^*)}{\partial p} + \lambda^* \frac{\partial G_1(z^*;\phi^*)}{\partial p} > 0$

Proof. When $f_v(v|\theta) = f_v(v)$ for all v we have that

$$\begin{aligned} \frac{\partial G_1(z^*;\phi^*)}{\partial p} &= [1 - F_v(p^*) - (p^* - c) f_v(p^*)] F_\theta\left(\frac{t^*}{\phi^*}\right) + \frac{t^*}{\phi^*}[1 - F_v(p^*)] f_\theta\left(\frac{t^*}{\phi^*}\right) \\ &= \frac{(p^* - c) f_v(p^*)}{\lambda^*} F_\theta\left(\frac{t^*}{\phi^*}\right) + \frac{t^*}{\phi^*}[1 - F_v(p^*)] f_\theta\left(\frac{t^*}{\phi^*}\right) \end{aligned}$$

where the second equality uses (35), the first order condition for p^* in the separable case. Next

$$\frac{\partial S_1(z^*;\phi^*)}{\partial p} = -(p^*-c) f_v(p^*) F_\theta\left(\frac{t^*}{\phi^*}\right) + \frac{t^*}{\phi^*}[1-F_v(p^*)] f_\theta\left(\frac{t^*}{\phi^*}\right)$$

Hence

$$\begin{aligned}\frac{\partial S_1(z^*;\phi^*)}{\partial p} + \lambda^*\frac{\partial G_1(z^*;\phi^*)}{\partial p} &= -(p^*-c) f_v(p^*) F_\theta\left(\frac{t^*}{\phi^*}\right) + \frac{t^*}{\phi^*}[1-F_v(p^*)] f_\theta\left(\frac{t^*}{\phi^*}\right) \\ &\quad +\lambda^*\left\{\frac{(p^*-c) f_v(p^*)}{\lambda^*} F_\theta\left(\frac{t^*}{\phi^*}\right) + \frac{t^*}{\phi^*}[1-F_v(p^*)] f_\theta\left(\frac{t^*}{\phi^*}\right)\right\} \\ &= (1+\lambda^*)\frac{t^*}{\phi^*}[1-F_v(p^*)] f_\theta\left(\frac{t^*}{\phi^*}\right) > 0.\end{aligned}$$

■

5 Genericity

This Section is Incomplete. Using the expressions for DG_1 and DS_1 in (44) and (46) respectively, we obtain:

$$\begin{aligned}&\frac{\partial S_1(z^*;\phi^*)}{\partial t} + \lambda^*\frac{\partial G_1(z^*;\phi^*)}{\partial t} \\ &= (1+\lambda^*)\left[\int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} (p^*-c) f_v(p^*|\theta) f_\theta(\theta)\, d\theta - \frac{t^*}{\phi^*} F_v\left(p^*|\frac{t^*}{\phi^*}\right) f_\theta\left(\frac{t^*}{\phi^*}\right)\right] \quad (55)\\ &+\lambda^*\int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} F_v(p^*|\theta) f_\theta(\theta)\, d\theta;\end{aligned}$$

$$\begin{aligned}&\frac{\partial S_1(z^*;\phi^*)}{\partial \tau} + \lambda^*\frac{\partial G_1(z^*;\phi^*)}{\partial \tau} \\ =\ & \lambda^*\int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} (1-F_v(p^*|\theta)) f_\theta(\theta)\, d\theta \quad (56)\\ &-(1+\lambda^*)\left[\int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} (p^*-c) f_v(p^*|\theta) f_\theta(\theta)\, d\theta + \frac{t^*}{\phi^*}\left[1-F_v\left(p^*|\frac{t^*}{\phi^*}\right)\right] f_\theta\left(\frac{t^*}{\phi^*}\right)\right]; \quad (57)\end{aligned}$$

$$\begin{aligned}&\frac{\partial S_1(z^*;\phi^*)}{\partial p} + \lambda^*\frac{\partial G_1(z^*;\phi^*)}{\partial p} \\ =\ & (1+\lambda^*)\left\{\frac{t^*}{\phi^*}\left[1-F_v\left(p^*|\frac{t^*}{\phi^*}\right)\right] f_\theta\left(\frac{t^*}{\phi^*}\right) - \int_{\underline{\theta}}^{\frac{t^*}{\phi^*}} (p^*-c) f_v(p^*|\theta) f_\theta(\theta)\, d\theta\right\} \quad (58)\\ &+\lambda^*\int_{\underline{\theta}}^{\frac{t^*}{\phi^*}} [1-F_v(p^*|\theta)] f_\theta(\theta)\, d\theta. \quad (59)\end{aligned}$$

We note that

$$\begin{aligned}
& \left[\frac{\partial S_1(z^*;\phi^*)}{\partial t}+\lambda^*\frac{\partial G_1(z^*;\phi^*)}{\partial t}\right]+\left[\frac{\partial S_1(z^*;\phi^*)}{\partial \tau}+\lambda^*\frac{\partial G_1(z^*;\phi^*)}{\partial \tau}\right] \\
= \ & -(1+\lambda^*)\frac{t^*}{\phi^*}f_\theta\left(\frac{t^*}{\phi^*}\right)+\lambda^*\left[1-F_\theta\left(\frac{t^*}{\phi^*}\right)\right]=0
\end{aligned}$$

as a result of the first order condition (34). Similarly,

$$\begin{aligned}
& \left[\frac{\partial S_1(z^*;\phi^*)}{\partial \tau}+\lambda^*\frac{\partial G_1(z^*;\phi^*)}{\partial \tau}\right]+\left[\frac{\partial S_1(z^*;\phi^*)}{\partial p}+\lambda^*\frac{\partial G_1(z^*;\phi^*)}{\partial p}\right] \\
= \ & \lambda^*\int_{\underline{\theta}}^{\bar{\theta}}\left[1-F_v(p^*|\theta)\right]f_\theta(\theta)\,d\theta-(1+\lambda^*)\int_{\underline{\theta}}^{\bar{\theta}}(p^*-c)\,f_v(p^*|\theta)\,f_\theta(\theta)\,d\theta \\
= \ & \lambda^*\left[1-F_v(p^*)\right]-(1+\lambda^*)(p^*-c)\,f_v(p^*)=0
\end{aligned}$$

as a result of the first order condition (35). Hence, the system $DS_1(z^*)+\lambda^*DG_1(z^*)\neq 0$ is actually equivalent to requiring one of (55), (57) and (59) to be non-zero. We are currently trying to verify that this condition is genericly satisfied.

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A Appendix: Omitted Calculations and Proofs

A.1 Derivation of Derivatives in (44):

For simplicity of notation, define

$$\begin{aligned}
A_1(z;\phi) &\equiv t\left[\int_{\frac{t}{\phi}}^{\overline{\theta}}\int_{\underline{v}}^{\tau-t} f(\theta,v)\,dvd\theta\right] \\
B_1(z;\phi) &\equiv (p-c)\left[\int_{p}^{\overline{v}}\int_{\underline{\theta}}^{\frac{\tau-p}{\phi^*}} f(\theta,v)\,d\theta dv\right] \\
C_1(z;\phi) &\equiv (\tau-c)\left[\int_{\frac{\tau-p}{\phi}}^{\frac{t}{\phi}}\int_{\tau-\phi\theta}^{\overline{v}} f(\theta,v)\,dvd\theta + \int_{\frac{t}{\phi}}^{\overline{\theta}}\int_{\tau-t}^{\overline{v}} f(\theta,v)\,dvd\theta\right]
\end{aligned}$$

so that

$$G_1(z;\phi) = A_1(z;\phi) + B_1(z;\phi) + C_1(z;\phi) - K\phi.$$

Differentiating with respect to t,

$$\begin{aligned}
\frac{\partial A_1(z;\phi)}{\partial t} &= \int_{\frac{t}{\phi}}^{\overline{\theta}}\int_{\underline{v}}^{\tau-t} f(\theta,v)\,dvd\theta - t\left[\int_{\underline{v}}^{\tau-t} f\left(\frac{t}{\phi},v\right)\frac{1}{\phi}dv + \int_{\frac{t}{\phi}}^{\overline{\theta}} f(\theta,\tau-t)\,d\theta\right], \\
\frac{\partial B_1(z;\phi)}{\partial t} &= 0, \\
\frac{\partial C_1(z;\phi)}{\partial t} &= (\tau-c)\left[\int_{\tau-t}^{\overline{v}} f\left(\frac{t}{\phi},v\right)\frac{1}{\phi}dv - \int_{\tau-t}^{\overline{v}} f\left(\frac{t}{\phi},v\right)\frac{1}{\phi}dv + \int_{\frac{t}{\phi}}^{\overline{\theta}} f(\theta,\tau-t)\,d\theta\right] \\
&= (\tau-c)\left[\int_{\frac{t}{\phi}}^{\overline{\theta}} f(\theta,\tau-t)\,d\theta\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{\partial A_1(z^*;\phi^*)}{\partial t} &= \int_{\frac{t^*}{\phi^*}}^{\overline{\theta}}\int_{\underline{v}}^{p^*} f(\theta,v)\,dvd\theta - t^*\left[\int_{\underline{v}}^{p^*} f\left(\frac{t^*}{\phi^*},v\right)\frac{1}{\phi^*}dv + \int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} f(\theta,p^*)\,d\theta\right] \\
\frac{\partial B_1(z^*;\phi^*)}{\partial t} &= 0 \\
\frac{\partial C_1(z^*;\phi^*)}{\partial t} &= (t^*+p^*-c)\left[\int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} f(\theta,p^*)\,d\theta\right].
\end{aligned}$$

So

$$\frac{\partial G_1(z^*)}{\partial t} = \int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} \left\{ \int_{\underline{v}}^{p^*} f(\theta, v)\, dv + (p^* - c) f(\theta, p^*) \right\} d\theta - t^* \int_{\underline{v}}^{p^*} f\left(\frac{t^*}{\phi^*}, v\right) \frac{1}{\phi^*} dv \tag{60}$$

$$= \int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} \left\{ \int_{\underline{v}}^{p^*} \underbrace{\frac{f(\theta, v)}{\int_{\underline{v}}^{\overline{v}} f(\theta, v)\, dv}}_{f_v(v|\theta)} dv + (p^* - c) \underbrace{\frac{f(\theta, p^*)}{\int_{\underline{v}}^{\overline{v}} f(\theta, v)\, dv}}_{f_v(p^*|\theta)} \right\} \underbrace{\left(\int_{\underline{v}}^{\overline{v}} f(\theta, v)\, dv \right)}_{f_\theta(\theta)} d\theta d\theta$$

$$-t^* \int_{\underline{v}}^{p^*} \underbrace{\frac{f\left(\frac{t^*}{\phi^*}, v\right)}{\int_{\underline{v}}^{\overline{v}} f\left(\frac{t^*}{\phi^*}, v\right) dv}}_{f_v\left(v|\frac{t^*}{\phi^*}\right)} \frac{1}{\phi^*} dv \underbrace{\int_{\underline{v}}^{\overline{v}} f\left(\frac{t^*}{\phi^*}, v\right) dv}_{f_\theta\left(\frac{t^*}{\phi^*}\right)}$$

$$= \int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} \left\{ \int_{\underline{v}}^{p^*} f_v(v|\theta)\, dv + (p^* - c) f_v(p^*|\theta) \right\} f_\theta(\theta)\, d\theta - \frac{t^*}{\phi^*} \int_{\underline{v}}^{p^*} f_v\left(v|\frac{t^*}{\phi^*}\right) dv f_\theta\left(\frac{t^*}{\phi^*}\right)$$

$$= \int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} \{F_v(p^*|\theta) + (p^* - c) f_v(p^*|\theta)\} f_\theta(\theta)\, d\theta - \frac{t^*}{\phi^*} F_v\left(p^*|\frac{t^*}{\phi^*}\right) f_\theta\left(\frac{t^*}{\phi^*}\right).$$

Differentiating with respect to τ yields

$$\frac{dA_1(z;\phi)}{d\tau} = t\left[\int_{\frac{t}{\phi}}^{\overline{\theta}} f(\theta, \tau - t)\, d\theta\right]$$

$$\frac{\partial B_1(z;\phi)}{\partial \tau} = (p - c)\left[\int_p^{\overline{v}} f\left(\frac{\tau - p}{\phi^*}, v\right) \frac{1}{\phi^*} dv\right]$$

$$\frac{\partial C_1(z;\phi)}{\partial \tau} = \int_{\frac{\tau-p}{\phi}}^{\frac{t}{\phi}} \int_{\tau-\phi\theta}^{\overline{v}} f(\theta, v)\, dv d\theta + \int_{\frac{t}{\phi}}^{\overline{\theta}} \int_{\tau-t}^{\overline{v}} f(\theta, v)\, dv d\theta$$
$$-(\tau - c)\left[\int_p^{\overline{v}} f\left(\frac{\tau - p}{\phi}, v\right) \frac{1}{\phi} dv + \int_{\frac{\tau-p}{\phi}}^{\frac{t}{\phi}} f(\theta, \tau - \phi\theta)\, d\theta\right]$$
$$-(\tau - c)\left[\int_{\frac{t}{\phi}}^{\overline{\theta}} f(\theta, \tau - t)\, d\theta\right]$$

Hence,

$$\frac{dA_1(z^*;\phi^*)}{d\tau} = t^*\left[\int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} f(\theta, \tau - t^*)\, d\theta\right]$$

$$\frac{\partial B_1(z^*;\phi^*)}{\partial \tau} = (p^* - c)\left[\int_{p^*}^{\overline{v}} f\left(\frac{t^*}{\phi^*}, v\right) \frac{1}{\phi^*} dv\right]$$

$$\frac{\partial C_1(z^*;\phi^*)}{\partial \tau} = \int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} \int_{p^*}^{\overline{v}} f(\theta, v)\, dv d\theta - (p^* + t^* - c)\left\{\left[\int_{p^*}^{\overline{v}} f\left(\frac{t^*}{\phi^*}, v\right) \frac{1}{\phi^*} dv\right] + \left[\int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} f(\theta, p^*)\, d\theta\right]\right\}.$$

It follows that

$$\begin{aligned}
\frac{\partial G_1(z^*;\phi^*)}{\partial \tau} &= \int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} \int_{p^*}^{\overline{v}} f(\theta, v)\, dv d\theta - t^* \left[\int_{p^*}^{\overline{v}} f\left(\frac{t^*}{\phi^*}, v\right) \frac{1}{\phi^*} dv\right] - (p^* - c)\left[\int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} f(\theta, p^*)\, dv d\theta\right] \\
&= \int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} \left\{\int_{p^*}^{\overline{v}} f(\theta, v)\, dv - (p^* - c) f(\theta, p^*)\right\} d\theta - t^* \left[\int_{p^*}^{\overline{v}} f\left(\frac{t^*}{\phi^*}, v\right) \frac{1}{\phi^*} dv\right] \\
\text{/same steps as in (60)/} \quad &= \int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} \{[1 - F_v(p^*|\theta)] - (p^* - c) f_v(p^*|\theta)\} f_\theta(\theta)\, d\theta - \frac{t^*}{\phi^*}\left[1 - F_v\left(p^*|\frac{t^*}{\phi^*}\right)\right] f_\theta\left(\frac{t^*}{\phi^*}\right).
\end{aligned}$$

Differentiating with respect to p yields

$$\begin{aligned}
\frac{\partial A_1(z;\phi)}{\partial p} &= 0 \\
\frac{\partial B_1(z;\phi)}{\partial p} &= \int_p^{\overline{v}} \int_{\underline{\theta}}^{\frac{\tau - p}{\phi}} f(\theta, v)\, d\theta dv - (p - c)\left[\int_{\underline{\theta}}^{\frac{\tau - p}{\phi}} f(\theta, p)\, d\theta + \int_p^{\overline{v}} f\left(\frac{\tau - p}{\phi}, v\right) \frac{1}{\phi} dv\right] \\
\frac{\partial C_1(z;\phi)}{\partial p} &= (\tau - c)\left[\int_p^{\overline{v}} f\left(\frac{\tau - p}{\phi}, v\right) \frac{1}{\phi} dv\right]
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\partial A_1(z^*;\phi^*)}{\partial p} &= 0 \\
\frac{\partial B_1(z^*;\phi^*)}{\partial p} &= \int_{p^*}^{\overline{v}} \int_{\underline{\theta}}^{\frac{t^*}{\phi^*}} f(\theta, v)\, d\theta dv - (p^* - c)\left[\int_{\underline{\theta}}^{\frac{t^*}{\phi^*}} f(\theta, p^*)\, d\theta + \int_{p^*}^{\overline{v}} f\left(\frac{t^*}{\phi^*}, v\right) \frac{1}{\phi^*} dv\right] \\
\frac{\partial C_1(z^*;\phi^*)}{\partial p} &= (t^* + p^* - c)\left[\int_{p^*}^{\overline{v}} f\left(\frac{t^*}{\phi^*}, v\right) \frac{1}{\phi^*} dv\right]
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial G_1(z^*;\phi^*)}{\partial p} &= \int_{\underline{\theta}}^{\frac{t^*}{\phi^*}} \left[\int_{p^*}^{\overline{v}} f(\theta, v)\, dv - (p^* - c) f(\theta, p^*)\right] d\theta + t^* \left[\int_{p^*}^{\overline{v}} f\left(\frac{t^*}{\phi^*}, v\right) \frac{1}{\phi^*} dv\right] \\
\text{/same steps as in (60)/} \quad &= \int_{\underline{\theta}}^{\frac{t^*}{\phi^*}} [\{(1 - F_v(p^*|\theta)) - (p^* - c) f_v(p^*|\theta)\}] f_\theta(\theta)\, d\theta \\
&\quad + \frac{t^*}{\phi^*}\left(1 - F_v\left(p^*|\frac{t^*}{\phi^*}\right)\right) f_\theta\left(\frac{t^*}{\phi^*}\right).
\end{aligned}$$

A.2 Derivation of Derivatives in (46):

Let

$$
\begin{aligned}
D_1(z;\phi) &= \int_{\frac{t}{\phi}}^{\overline{\theta}} \int_{\underline{v}}^{\tau-t} \phi\theta f(\theta,v)\,dvd\theta \\
E_1(z;\phi) &= \int_{p}^{\overline{v}} \int_{\underline{\theta}}^{\frac{\tau-p}{\phi}} (v-c) f(\theta,v)\,d\theta dv \\
F_1(z;\phi) &= \int_{\frac{\tau-p}{\phi}}^{\frac{t}{\phi}} \int_{\tau-\phi\theta}^{\overline{v}} (\phi\theta+v-c) f(\theta,v)\,dvd\theta + \int_{\frac{t}{\phi}}^{\overline{\theta}} \int_{\tau-t}^{\overline{v}} (\phi\theta+v-c) f(\theta,v)\,dvd\theta
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\frac{\partial D_1(z;\phi)}{\partial t} &= -\int_{\underline{v}}^{\tau-t} tf\left(\frac{t}{\phi},v\right)\frac{1}{\phi}dv - \int_{\frac{t}{\phi}}^{\overline{\theta}} \phi\theta f(\theta,\tau-t)\,d\theta \\
\frac{\partial D_1(z;\phi)}{\partial p} &= 0 \\
\frac{\partial D_1(z;\phi)}{\partial \tau} &= \int_{\frac{t}{\phi}}^{\overline{\theta}} \phi\theta f(\theta,\tau-t)\,d\theta
\end{aligned}
$$

and thus,

$$
\begin{aligned}
\frac{\partial D_1(z^*;\phi^*)}{\partial t} &= -\int_{\underline{v}}^{p^*} t^* f\left(\frac{t^*}{\phi^*},v\right)\frac{1}{\phi^*}dv - \int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} \phi^*\theta f(\theta,p^*)\,d\theta \\
\frac{\partial D_1(z^*;\phi^*)}{\partial p} &= 0 \\
\frac{\partial D_1(z^*;\phi^*)}{\partial \tau} &= \int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} \phi^*\theta f(\theta,p^*)\,dvd\theta.
\end{aligned}
\tag{61}
$$

Similarly,

$$
\begin{aligned}
\frac{\partial E_1(z;\phi)}{\partial t} &= 0 \\
\frac{\partial E_1(z;\phi)}{\partial p} &= -\int_{\underline{\theta}}^{\frac{\tau-p}{\phi}} (p-c) f(\theta,p)\,d\theta - \int_{p}^{\overline{v}} (v-c) f\left(\frac{\tau-p}{\phi},v\right)\frac{1}{\phi}dv \\
\frac{\partial E_1(z;\phi)}{\partial \tau} &= \int_{p}^{\overline{v}} (v-c) f\left(\frac{\tau-p}{\phi},v\right)\frac{1}{\phi}dv
\end{aligned}
$$

thus,

$$
\begin{aligned}
\frac{\partial E_1(z^*;\phi^*)}{\partial t} &= 0 \\
\frac{\partial E_1(z^*;\phi^*)}{\partial p} &= -\int_{\underline{\theta}}^{\frac{t^*}{\phi^*}} (p^*-c) f(\theta,p^*)\,d\theta - \int_{p^*}^{\overline{v}} (v-c) f\left(\frac{t^*}{\phi^*},v\right)\frac{1}{\phi^*}dv \\
\frac{\partial E_1(z^*;\phi^*)}{\partial \tau} &= \int_{p^*}^{\overline{v}} (v-c) f\left(\frac{t^*}{\phi^*},v\right)\frac{1}{\phi^*}dv.
\end{aligned}
\tag{62}
$$

Finally,

$$
\begin{aligned}
\frac{\partial F_1(z;\phi)}{\partial t} &= \int_{\tau-t}^{\overline{v}} (t+v-c)\, f\left(\frac{t}{\phi}, v\right)\frac{1}{\phi}dv - \int_{\tau-t}^{\overline{v}} (t+v-c)\, f\left(\frac{t}{\phi}, v\right)\frac{1}{\phi}dv \\
&\quad + \int_{\frac{t}{\phi}}^{\overline{\theta}} (\phi\theta+\tau-t-c)\, f(\theta,\tau-t)\, d\theta \\
&= \int_{\frac{t}{\phi}}^{\overline{\theta}} (\phi\theta+\tau-t-c)\, f(\theta,\tau-t) \\
\frac{\partial F_1(z;\phi)}{\partial p} &= \int_{p}^{\overline{v}} (\tau-p+v-c)\, f\left(\frac{\tau-p}{\phi}, v\right)\frac{1}{\phi}dv \\
\frac{\partial F_1(z;\phi)}{\partial \tau} &= -\int_{p}^{\overline{v}} (\tau-p+v-c)\, f\left(\frac{\tau-p}{\phi}, v\right)\frac{1}{\phi}dv - \int_{\frac{\tau-p}{\phi}}^{\frac{t}{\phi}} (\tau-c)\, f(\theta,\tau-\phi\theta)\, d\theta \\
&\quad - \int_{\frac{t}{\phi}}^{\overline{\theta}} (\phi\theta+\tau-t-c)\, f(\theta,\tau-t)\, d\theta
\end{aligned}
$$

thus,

$$
\begin{aligned}
\frac{\partial F_1(z^*;\phi^*)}{\partial t} &= \int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} (\phi^*\theta+p^*-c)\, f(\theta,p^*)\, d\theta \qquad (63)\\
\frac{\partial F_1(z^*;\phi^*)}{\partial p} &= \int_{p^*}^{\overline{v}} (t^*+v-c)\, f\left(\frac{t^*}{\phi^*}, v\right)\frac{1}{\phi^*}dv \\
\frac{\partial F_1(z^*;\phi^*)}{\partial \tau} &= -\int_{p^*}^{\overline{v}} (t^*+v-c)\, f\left(\frac{t^*}{\phi^*}, v\right)\frac{1}{\phi^*}dv - \int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} (\phi^*\theta+p^*-c)\, f(\theta,p^*)\, d\theta.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{\partial S_1(z^*;\phi^*)}{\partial t} &= \frac{\partial D_1(z^*;\phi^*)}{\partial t} + \frac{\partial E_1(z^*;\phi^*)}{\partial t} + \frac{\partial F_1(z^*;\phi^*)}{\partial t} \\
&= -\int_{\underline{v}}^{p^*} t^* f\left(\frac{t^*}{\phi^*}, v\right)\frac{1}{\phi^*}dv - \int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} \phi^*\theta f(\theta,p^*)\, d\theta + \int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} (\phi^*\theta+p^*-c)\, f(\theta,p^*)\, d\theta \\
&= -\int_{\underline{v}}^{p^*} \frac{t^*}{\phi^*} f\left(\frac{t^*}{\phi^*}, v\right) dv + \int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} (p^*-c)\, f(\theta,p^*)\, d\theta \\
&= \int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} (p^*-c)\, f_v(p^*|\theta)\, f_\theta(\theta)\, d\theta - \frac{t^*}{\phi^*} F_v\left(p^*|\frac{t^*}{\phi^*}\right) f_\theta\left(\frac{t^*}{\phi^*}\right)
\end{aligned}
$$

and,

$$\begin{aligned}
\frac{\partial S_1(z^*;\phi^*)}{\partial p} &= \frac{\partial D_1(z^*;\phi^*)}{\partial p} + \frac{\partial E_1(z^*;\phi^*)}{\partial p} + \frac{\partial F_1(z^*;\phi^*)}{\partial p} \\
&= -\int_{\underline{\theta}}^{\frac{t^*}{\phi}} (p^* - c) f(\theta, p^*) d\theta - \int_{p^*}^{\overline{v}} (v - c) f\left(\frac{t^*}{\phi^*}, v\right) \frac{1}{\phi^*} dv \\
&\quad + \int_{p^*}^{\overline{v}} (t^* + v - c) f\left(\frac{t^*}{\phi^*}, v\right) \frac{1}{\phi^*} dv \\
&= -\int_{\underline{\theta}}^{\frac{t^*}{\phi}} (p^* - c) f(\theta, p^*) d\theta + \int_{p^*}^{\overline{v}} \frac{t^*}{\phi^*} f\left(\frac{t^*}{\phi^*}, v\right) dv \\
&= -\int_{\underline{\theta}}^{\frac{t^*}{\phi}} (p^* - c) f_v(p^*|\theta) f_\theta(\theta) d\theta + \frac{t^*}{\phi^*}\left[1 - F_v\left(p^*|\frac{t^*}{\phi^*}\right)\right] f_\theta\left(\frac{t^*}{\phi^*}\right)
\end{aligned}$$

and,

$$\begin{aligned}
\frac{\partial S_1(z^*;\phi^*)}{\partial \tau} &= \frac{\partial D_1(z^*;\phi^*)}{\partial \tau} + \frac{\partial E_1(z^*;\phi^*)}{\partial \tau} + \frac{\partial F_1(z^*;\phi^*)}{\partial \tau} \\
&= \int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} \phi^* \theta f(\theta, p^*) dv d\theta + \int_{p^*}^{\overline{v}} (v - c) f\left(\frac{t^*}{\phi^*}, v\right) \frac{1}{\phi^*} dv \\
&\quad - \int_{p^*}^{\overline{v}} (t^* + v - c) f\left(\frac{t^*}{\phi^*}, v\right) \frac{1}{\phi^*} dv - \int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} (\phi^*\theta + p^* - c) f(\theta, p^*) d\theta \\
&= -\int_{p^*}^{\overline{v}} \frac{t^*}{\phi^*} f\left(\frac{t^*}{\phi^*}, v\right) dv - \int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} (p^* - c) f(\theta, p^*) d\theta \\
&= -\int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} (p^* - c) f_v(p^*|\theta) f_\theta(\theta) d\theta - \frac{t^*}{\phi^*}\left[1 - F_v\left(p^*|\frac{t^*}{\phi^*}\right)\right] f_\theta\left(\frac{t^*}{\phi^*}\right).
\end{aligned}$$

A.3 Calculations in Proving Lemma 4

- **Part 1:** $DG_1(z^*;\phi^*) = DG_2(z^*;\phi^*)$:

Write $G_2(z;\phi) = A_2(z;\phi) + B_2(z;\phi) + C_2(z;\phi) - K\phi$ where:

$$
\begin{aligned}
A_2(z;\phi) &\equiv t\left[\int_{\frac{t}{\phi}}^{\frac{\tau-p}{\phi}}\int_{\underline{v}}^{\phi\theta+p-t} f(\theta,v)\,dvd\theta + \int_{\frac{\tau-p}{\phi}}^{\overline{\theta}}\int_{\underline{v}}^{\tau-t} f(\theta,v)\,dvd\theta\right] \\
B_2(z;\phi) &\equiv (p-c)\left[\int_{p}^{\tau-t}\int_{\underline{\theta}}^{\frac{v+t-p}{\phi}} f(\theta,v)\,d\theta dv + \int_{\tau-t}^{\overline{v}}\int_{\underline{\theta}}^{\frac{\tau-p}{\phi}} f(\theta,v)\,d\theta dv\right] \\
C_2(z;\phi) &\equiv (\tau-c)\left[\int_{\frac{\tau-p}{\phi}}^{\overline{\theta}}\int_{\tau-t}^{\overline{v}} f(\theta,v)\,dvd\theta\right].
\end{aligned}
$$

Differentiating A_2 we get:

$$
\begin{aligned}
\frac{\partial A_2(z;\phi)}{\partial t} &= \left[\int_{\frac{t}{\phi}}^{\frac{\tau-p}{\phi}}\int_{\underline{v}}^{\phi\theta+p-t} f(\theta,v)\,dvd\theta + \int_{\frac{\tau-p}{\phi}}^{\overline{\theta}}\int_{\underline{v}}^{\tau-t} f(\theta,v)\,dvd\theta\right] \\
&\quad -t\left[\int_{\underline{v}}^{p} f\left(\frac{t}{\phi},v\right)\frac{1}{\phi}dv + \int_{\frac{t}{\phi}}^{\frac{\tau-p}{\phi}} f(\theta,\phi\theta+p-t)\,d\theta + \int_{\frac{\tau-p}{\phi}}^{\overline{\theta}} f(\theta,\tau-t)\,d\theta\right] \\
\frac{\partial A_2(z;\phi)}{\partial p} &= -t\int_{\underline{v}}^{\tau-t} f\left(\frac{\tau-p}{\phi},v\right)\frac{1}{\phi}dv + t\int_{\frac{t}{\phi}}^{\frac{\tau-p}{\phi}} f(\theta,\phi\theta+p-t)\,d\theta + t\int_{\underline{v}}^{\tau-t} f\left(\frac{\tau-p}{\phi},v\right)\frac{1}{\phi}dv \\
&= t\int_{\frac{t}{\phi}}^{\frac{\tau-p}{\phi}} f(\theta,\phi\theta+p-t)\,d\theta \\
\frac{\partial A_2(z;\phi)}{\partial \tau} &= t\int_{\underline{v}}^{\tau-t} f\left(\frac{\tau-p}{\phi},v\right)\frac{1}{\phi}dv - t\int_{\underline{v}}^{\tau-t} f\left(\frac{\tau-p}{\phi},v\right)\frac{1}{\phi}dv + t\int_{\frac{\tau-p}{\phi}}^{\overline{\theta}} f(\theta,\tau-t)\,d\theta \\
&= t\int_{\frac{\tau-p}{\phi}}^{\overline{\theta}} f(\theta,\tau-t)\,d
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{\partial A_2(z^*;\phi^*)}{\partial t} &= \int_{\frac{t^*}{\phi^*}}^{\overline{\theta}}\int_{\underline{v}}^{p^*} f(\theta,v)\,dvd\theta - t^*\left[\int_{\underline{v}}^{p^*} f\left(\frac{t^*}{\phi^*},v\right)\frac{1}{\phi^*}dv + \int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} f(\theta,p^*)\,d\theta\right] \\
\frac{\partial A_2(z^*;\phi^*)}{\partial p} &= 0 \\
\frac{\partial A_2(z^*;\phi^*)}{\partial \tau} &= t^*\int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} f(\theta,p^*)\,d\theta.
\end{aligned}
$$

Similarly,

$$\begin{aligned}
\frac{\partial B_2(z;\phi)}{\partial t} &= (p-c)\left[-\int_{\underline{\theta}}^{\frac{\tau-p}{\phi}} f(\theta,\tau-t)\,d\theta + \int_p^{\tau-t} f\left(\frac{v+t-p}{\phi},v\right)\frac{1}{\phi}dv + \int_{\underline{\theta}}^{\frac{\tau-p}{\phi}} f(\theta,\tau-t)\,d\theta\right] \\
&= (p-c)\left[\int_p^{\tau-t} f\left(\frac{v+t-p}{\phi},v\right)\frac{1}{\phi}dv\right] \\
\frac{\partial B_2(z;\phi)}{\partial p} &= \int_p^{\tau-t}\int_{\underline{\theta}}^{\frac{v+t-p}{\phi}} f(\theta,v)\,d\theta dv + \int_{\tau-t}^{\overline{v}}\int_{\underline{\theta}}^{\frac{\tau-p}{\phi}} f(\theta,v)\,dv d\theta \\
&\quad -(p-c)\left[\int_{\underline{\theta}}^{\frac{t}{\phi}} f(\theta,p)\,d\theta + \int_p^{\tau-t} f\left(\frac{v+t-p}{\phi},v\right)\frac{1}{\phi}dv + \int_{\tau-t}^{\overline{v}} f\left(\frac{\tau-p}{\phi},v\right)dv\right] \\
\frac{\partial B_2(z;\phi)}{\partial \tau} &= (p-c)\left[\int_{\underline{\theta}}^{\frac{\tau-p}{\phi}} f(\theta,\tau-t)\,d\theta dv - \int_{\underline{\theta}}^{\frac{\tau-p}{\phi}} f(\theta,\tau-t)\,d\theta + \int_{\tau-t}^{\overline{v}} f\left(\frac{\tau-p}{\phi},v\right)\frac{1}{\phi}d\theta\right] \\
&= (p-c)\left[\int_{\tau-t}^{\overline{v}} f\left(\frac{\tau-p}{\phi},v\right)\frac{1}{\phi}d\theta\right]
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\partial B_2(z^*;\phi^*)}{\partial t} &= 0 \\
\frac{\partial B_2(z^*;\phi^*)}{\partial p} &= \int_{p^*}^{\overline{v}}\int_{\underline{\theta}}^{\frac{t^*}{\phi^*}} f(\theta,v)\,dv d\theta - (p^*-c)\left[\int_{\underline{\theta}}^{\frac{t^*}{\phi^*}} f(\theta,p^*)\,d\theta + \int_{p^*}^{\overline{v}} f\left(\frac{t^*}{\phi^*},v\right)dv\right] \\
\frac{\partial B_2(z^*;\phi^*)}{\partial \tau} &= (p^*-c)\left[\int_{p^*}^{\overline{v}} f\left(\frac{t^*}{\phi^*},v\right)\frac{1}{\phi^*}d\theta\right].
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\partial C_2(z;\phi)}{\partial t} &= (\tau-c)\left[\int_{\frac{\tau-p}{\phi}}^{\overline{\theta}} f(\theta,\tau-t)\,d\theta\right] \\
\frac{\partial C_2(z;\phi)}{\partial p} &= (\tau-c)\left[\int_{\tau-t}^{\overline{v}} f\left(\frac{\tau-p}{\phi},v\right)\frac{1}{\phi}dv\right] \\
\frac{\partial C_2(z;\phi)}{\partial \tau} &= \int_{\frac{\tau-p}{\phi}}^{\overline{\theta}}\int_{\tau-t}^{\overline{v}} f(\theta,v)\,dv d\theta - (\tau-c)\left[\int_{\tau-t}^{\overline{v}} f\left(\frac{\tau-p}{\phi},v\right)\frac{1}{\phi}dv + \int_{\frac{\tau-p}{\phi}}^{\overline{\theta}} f(\theta,\tau-t)\,d\theta\right]
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\partial C_2(z^*;\phi^*)}{\partial t} &= (t^*+p^*-c)\left[\int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} f(\theta,p^*)\,d\theta\right] \\
\frac{\partial C_2(z^*;\phi^*)}{\partial p} &= (t^*+p^*-c)\left[\int_{p^*}^{\overline{v}} f\left(\frac{t^*}{\phi^*},v\right)\frac{1}{\phi^*}dv\right] \\
\frac{\partial C_2(z^*;\phi^*)}{\partial \tau} &= \int_{\frac{t^*}{\phi^*}}^{\overline{\theta}}\int_{p^*}^{\overline{v}} f(\theta,v)\,dv d\theta - (t^*+p^*-c)\left[\int_{p^*}^{\overline{v}} f\left(\frac{t^*}{\phi^*},v\right)\frac{1}{\phi^*}dv + \int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} f(\theta,p^*)\,d\theta\right]
\end{aligned}$$

Combining terms we get that

$$\begin{aligned}
\frac{\partial G_2(z^*;\phi^*)}{\partial t} &= \frac{\partial A_2(z^*)}{\partial t}+\frac{\partial B_2(z^*)}{\partial t}+\frac{\partial C_2(z^*)}{\partial t} \\
&= \int_{\frac{t^*}{\phi^*}}^{\overline{\theta}}\int_{\underline{v}}^{p^*} f(\theta,v)\,dv d\theta - t^*\left[\int_{\underline{v}}^{p^*} f\left(\frac{t^*}{\phi^*},v\right)\frac{1}{\phi^*}dv+\int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} f(\theta,p^*)\,d\theta\right] \\
&\quad +(t^*+p^*-c)\left[\int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} f(\theta,p^*)\,d\theta\right] \\
&= \int_{\frac{t^*}{\phi^*}}^{\overline{\theta}}\left\{\int_{\underline{v}}^{p^*} f(\theta,v)\,dv+(p^*-c)\,f(\theta,p^*)\right\}d\theta - t^*\int_{\underline{v}}^{p^*} f\left(\frac{t^*}{\phi^*},v\right)\frac{1}{\phi^*}dv \\
&= \int_{\frac{t^*}{\phi^*}}^{\overline{\theta}}\left\{\int_{\underline{v}}^{p^*} f_v(v|\theta)\,dv+(p^*-c)\,f_v(p^*|\theta)\right\}f_\theta(\theta)\,d\theta - \frac{t^*}{\phi^*}\int_{\underline{v}}^{p^*} f_v\left(v|\frac{t^*}{\phi^*}\right)dv f_\theta\left(\frac{t^*}{\phi^*}\right) \\
&= \int_{\frac{t^*}{\phi^*}}^{\overline{\theta}}\left\{F_v(p^*|\theta)+(p^*-c)\,f_v(p^*|\theta)\right\}f_\theta(\theta)\,d\theta - \frac{t^*}{\phi^*}F_v\left(p^*|\frac{t^*}{\phi^*}\right)f_\theta\left(\frac{t^*}{\phi^*}\right) \\
&= \frac{\partial G_1(z^*)}{\partial t}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial G_2(z^*;\phi^*)}{\partial p} &= \frac{\partial A_2(z^*;\phi^*)}{\partial p}+\frac{\partial B_2(z^*;\phi^*)}{\partial p}+\frac{\partial C_2(z^*;\phi^*)}{\partial p} \\
&= \int_{p^*}^{\overline{v}}\int_{\underline{\theta}}^{\frac{t^*}{\phi^*}} f(\theta,v)\,d\theta dv - (p^*-c)\left[\int_{\underline{\theta}}^{\frac{t^*}{\phi^*}} f(\theta,p^*)\,d\theta+\int_{p^*}^{\overline{v}} f\left(\frac{t^*}{\phi^*},v\right)dv\right] \\
&\quad +(t^*+p^*-c)\left[\int_{p^*}^{\overline{v}} f\left(\frac{t^*}{\phi^*},v\right)\frac{1}{\phi^*}dv\right] \\
&= \int_{\underline{\theta}}^{\frac{t^*}{\phi^*}}\left\{\int_{p^*}^{\overline{v}} f(\theta,v)\,dv-(p^*-c)\,f(\theta,p^*)\right\}d\theta + t^*\left[\int_{p^*}^{\overline{v}} f\left(\frac{t^*}{\phi^*},v\right)\frac{1}{\phi^*}dv\right] \\
&= \int_{\underline{\theta}}^{\frac{t^*}{\phi^*}}\left\{1-F_v(p^*|\theta)-(p^*-c)\,f_v(p^*|\theta)\right\}f_\theta(\theta)\,d\theta + t^*\left[1-F_v\left(p^*|\frac{t^*}{\phi^*}\right)\right]f_\theta\left(\frac{t^*}{\phi^*}\right) \\
&= \frac{\partial G_1(z^*)}{\partial p}
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial G_2(z^*;\phi^*)}{\partial\tau} &= \frac{\partial A_2(z^*;\phi^*)}{\partial\tau}+\frac{\partial B_2(z^*;\phi^*)}{\partial\tau}+\frac{\partial C_2(z^*;\phi^*)}{\partial\tau}\\
&= t^*\int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} f(\theta,p^*)\,d\theta+(p^*-c)\left[\int_{p^*}^{\overline{v}} f\left(\frac{t^*}{\phi^*},v\right)\frac{1}{\phi^*}d\theta\right]\\
&\quad+\int_{\frac{t^*}{\phi^*}}^{\overline{\theta}}\int_{p^*}^{\overline{v}} f(\theta,v)\,dvd\theta-(t^*+p^*-c)\left[\int_{p^*}^{\overline{v}} f\left(\frac{t^*}{\phi^*},v\right)\frac{1}{\phi^*}dv+\int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} f(\theta,p^*)\,d\theta\right]\\
&= \int_{\frac{t^*}{\phi^*}}^{\overline{\theta}}\left\{\int_{p^*}^{\overline{v}} f(\theta,v)\,dv-(p^*-c)\,f(\theta,p^*)\right\}d\theta-t^*\left[\int_{p^*}^{\overline{v}} f\left(\frac{t^*}{\phi^*},v\right)\frac{1}{\phi^*}dv\right]\\
&= \int_{\frac{t^*}{\phi^*}}^{\overline{\theta}}\{1-F_v(p^*|\theta)-(p^*-c)\,f_v(p^*|\theta)\}\,f_\theta(\theta)\,d\theta-\frac{t^*}{\phi^*}\left[1-F_v\left(p^*|\frac{t^*}{\phi^*}\right)f_\theta\left(\frac{t^*}{\phi^*}\right)\right]\\
&= \frac{\partial G_1(z^*)}{\partial\tau}
\end{aligned}
$$

- **Part 2:** $DS_1(z^*;\phi^*)=DS_2(z^*;\phi^*)$**:**

Write $S_2(z;\phi)=D_2(z;\phi)+E_2(z;\phi)+F_2(z;\phi)-K\phi$ where

$$
\begin{aligned}
D_2(z;\phi) &= \int_{\frac{t}{\phi}}^{\frac{\tau-p}{\phi}}\int_{\underline{v}}^{\phi\theta+p-t}\phi\theta f(\theta,v)\,dvd\theta+\int_{\frac{\tau-p}{\phi}}^{\overline{\theta}}\int_{\underline{v}}^{\tau-t}\phi\theta f(\theta,v)\,dvd\theta\\
E_2(z;\phi) &= \int_{p}^{\tau-t}\int_{\underline{\theta}}^{\frac{v+t-p}{\phi}}(v-c)\,f(\theta,v)\,d\theta dv+\int_{\tau-t}^{\overline{v}}\int_{\underline{\theta}}^{\frac{\tau-p}{\phi}}(v-c)\,f(\theta,v)\,d\theta dv\\
F_2(z;\phi) &= \int_{\frac{\tau-p}{\phi}}^{\overline{\theta}}\int_{\tau-t}^{\overline{v}}(\phi\theta+v-c)\,f(\theta,v)\,dvd\theta.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\frac{\partial D_2(z;\phi)}{\partial t} &= -\int_{\underline{v}}^{p}\frac{t}{\phi}f\left(\frac{t}{\phi},v\right)dv-\int_{\frac{t}{\phi}}^{\frac{\tau-p}{\phi}}\phi\theta f(\theta,\phi\theta+p-t)\,d\theta-\int_{\frac{\tau-p}{\phi}}^{\overline{\theta}}\phi\theta f(\theta,\tau-t)\,d\theta\\
\frac{\partial D_2(z;\phi)}{\partial p} &= -\int_{\underline{v}}^{\tau-t}\frac{\tau-p}{\phi}f\left(\frac{\tau-p}{\phi},v\right)dv+\int_{\frac{t}{\phi}}^{\frac{\tau-p}{\phi}}\phi\theta f(\theta,\phi\theta+p-t)\,dvd\theta+\int_{\underline{v}}^{\tau-t}\frac{\tau-p}{\phi}f\left(\frac{\tau-p}{\phi},v\right)dv\\
&= \int_{\frac{t}{\phi}}^{\frac{\tau-p}{\phi}}\phi\theta f(\theta,\phi\theta+p-t)\,dvd\theta\\
\frac{\partial D_2(z;\phi)}{\partial\tau} &= \int_{\underline{v}}^{\tau-t}\frac{\tau-p}{\phi}f\left(\frac{\tau-p}{\phi},v\right)dv-\int_{\underline{v}}^{\tau-t}\frac{\tau-p}{\phi}f\left(\frac{\tau-p}{\phi},v\right)dv+\int_{\frac{\tau-p}{\phi}}^{\overline{\theta}}\phi\theta f(\theta,\tau-t)\,d\theta\\
&= \int_{\frac{\tau-p}{\phi}}^{\overline{\theta}}\phi\theta f(\theta,\tau-t)\,d\theta
\end{aligned}
$$

so

$$\begin{aligned}
\frac{\partial D_2(z^*;\phi^*)}{\partial t} &= -\int_{\underline{v}}^{p^*} \frac{t^*}{\phi^*} f\left(\frac{t^*}{\phi^*}, v\right) dv - \int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} \phi^* \theta f(\theta, p^*)\, d\theta \\
\frac{\partial D_2(z^*;\phi^*)}{\partial p} &= 0 \\
\frac{\partial D_2(z^*;\phi^*)}{\partial \tau} &= \int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} \phi^* \theta f(\theta, p^*)\, d\theta,
\end{aligned}$$

which is the same as in (61).

Similarly,

$$\begin{aligned}
\frac{\partial E_2(z;\phi)}{\partial t} &= -\int_{\underline{\theta}}^{\frac{\tau-p}{\phi}} (\tau - t - c) f(\theta, \tau - t)\, d\theta + \int_{p}^{\tau-t} (v-c) f\left(\frac{v+t-p}{\phi}, v\right) \frac{1}{\phi} dv \\
&\quad + \int_{\underline{\theta}}^{\frac{\tau-p}{\phi}} (\tau - t - c) f(\theta, \tau - t)\, d\theta \\
&= \int_{p}^{\tau-t} (v-c) f\left(\frac{v+t-p}{\phi}, v\right) \frac{1}{\phi} dv \\
\frac{\partial E_2(z;\phi)}{\partial p} &= -\int_{\underline{\theta}}^{\frac{t}{\phi}} (p-c) f(\theta, p)\, d\theta - \int_{p}^{\tau-t} (v-c) f\left(\frac{v+t-p}{\phi}, v\right) \frac{1}{\phi} dv \\
&\quad - \int_{\tau-t}^{\overline{v}} (v-c) f\left(\frac{\tau-p}{\phi}, v\right) \frac{1}{\phi} dv \\
\frac{\partial E_2(z;\phi)}{\partial \tau} &= \int_{\underline{\theta}}^{\frac{\tau-p}{\phi}} (\tau - t - c) f(\theta, \tau - t)\, d\theta - \int_{\underline{\theta}}^{\frac{\tau-p}{\phi}} (\tau - t - c) f(\theta, \tau - t)\, d\theta \\
&\quad + \int_{\tau-t}^{\overline{v}} (v-c) f\left(\frac{\tau-p}{\phi}, v\right) \frac{1}{\phi} dv \\
&= \int_{\tau-t}^{\overline{v}} (v-c) f\left(\frac{\tau-p}{\phi}, v\right) \frac{1}{\phi} dv
\end{aligned}$$

so,

$$\begin{aligned}
\frac{\partial E_2(z^*;\phi^*)}{\partial t} &= 0 \\
\frac{\partial E_2(z^*;\phi^*)}{\partial p} &= -\int_{\underline{\theta}}^{\frac{t^*}{\phi^*}} (p^* - c) f(\theta, p^*)\, d\theta - \int_{p^*}^{\overline{v}} (v-c) f\left(\frac{t^*}{\phi^*}, v\right) \frac{1}{\phi^*} dv \\
\frac{\partial E_2(z^*;\phi^*)}{\partial \tau} &= \int_{p^*}^{\overline{v}} (v-c) f\left(\frac{t^*}{\phi^*}, v\right) \frac{1}{\phi^*} dv,
\end{aligned}$$

which is the same as the expressions in (62).

Finally,

$$\begin{aligned}
\frac{\partial F_2(z;\phi)}{\partial t} &= \int_{\frac{\tau-p}{\phi}}^{\overline{\theta}} (\phi\theta+\tau-t-c)\, f(\theta,\tau-t)\, d\theta \\
\frac{\partial F_2(z;\phi)}{\partial p} &= \int_{\tau-t}^{\overline{v}} (\tau-p+v-c)\, f\left(\frac{\tau-p}{\phi},v\right)\frac{1}{\phi}dv \\
\frac{\partial F_2(z;\phi)}{\partial \tau} &= -\int_{\tau-t}^{\overline{v}} (\tau-p+v-c)\, f\left(\frac{\tau-p}{\phi},v\right)\frac{1}{\phi}dv - \int_{\frac{\tau-p}{\phi}}^{\overline{\theta}} (\phi\theta+\tau-t-c)\, f(\theta,\tau-t)\, d\theta
\end{aligned}$$

so,

$$\begin{aligned}
\frac{\partial F_2(z^*;\phi^*)}{\partial t} &= \int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} (\phi^*\theta+p^*-c)\, f(\theta,p^*)\, d\theta \\
\frac{\partial F_2(z^*;\phi^*)}{\partial p} &= \int_{p^*}^{\overline{v}} (t^*+v-c)\, f\left(\frac{t^*}{\phi^*},v\right)\frac{1}{\phi^*}dv \\
\frac{\partial F_2(z^*;\phi^*)}{\partial \tau} &= -\int_{p^*}^{\overline{v}} (t^*+v-c)\, f\left(\frac{t^*}{\phi^*},v\right)\frac{1}{\phi^*}dv - \int_{\frac{t^*}{\phi^*}}^{\overline{\theta}} [\phi^*\theta+p^*-c]\, f(\theta,p^*)\, d\theta
\end{aligned}$$

which is the same as the expressions in (63). Since all the components are identical the result follows. ■